Reading 8

H_0, the circle, graphs

Today we will compute the homology of the circle, and some graphs.

8.1 Short exact sequences and a proposition about very short exact sequences

This section is new as of February 15

Let's get some more practice with this notion of "exactness" of a sequence.

Definition 8.1.1. A *short exact sequence* is an exact sequence of group homomorphisms (between abelian groups) of the form

$$0 \to A \to B \to C \to 0.$$

Proposition 8.1.2. Any short exact sequence induces an isomorphism $B/A \cong C$.

Remark 8.1.3. Let us be more precise with the claim – after all, A is not a subgroup of B, so there is no way to quotient B by A.

Let us label the homomorphisms as follows:

$$0 \xrightarrow{z} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{z'} 0 \tag{8.1.0.1}$$

By exactness, $\text{im}(z) = \ker(f)$; because the domain of z is the zero abelian group, the image of z is trivial (i.e., the image consists of the 0 element of A). Thus $\ker(f)$ is trivial, so f is an injection.

In this way, we can identify A with a subgroup of B by identifying A with $f(A)$. (This is only possible because f is an injection, of course.) So when we write B/A, one really means $B/f(A)$.

Regardless, the notation B/A is very common in the literature. Now you know what it actually means.

Proof of Proposition 8.1.2. We follow the notation from (8.1.0.1). We have already seen that f is an injection.

On the other hand, because z' is the map to the zero group, $\ker(z') = C$. By exactness, we know that $\ker(z') = \mathrm{im}(g)$. Thus, by the first isomorphism theorem, g induces an isomorphism

$$B/\ker(g) \cong C.$$

By exactness, we know $\ker(g) = \mathrm{im}(f)$, and the claim follows. □

Example 8.1.4. The sequence of maps

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

– where f takes the element $[1] \in \mathbb{Z}/2\mathbb{Z}$ to the element $[2] \in \mathbb{Z}/4\mathbb{Z}$ and g is the unique onto homomorphism – is a short exact sequence. Note that $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so short exact sequences can be interesting: They do not just encode direct sums.

You might have wondered why the above kinds of sequences are called "short" exact sequences – surely, there are shorter exact sequences! Well, any exact sequence of a shorter length is much less interesting:

Proposition 8.1.5. If $0 \to A \to 0$ is an exact sequence, then A is (isomorphic to) the zero group.

If $0 \to A \to B \to 0$ is an exact sequence, then the map $A \to B$ is an isomorphism.

Proof. For the first claim, let us label the maps $z : 0 \to A$ and $z' : A \to 0$. Then the image of z is the kernel of z' by exactness. The image of z is trivial (because the domain of z is the zero group), so z' is an injection. On the other hand, z' is a surjection (because the codomain of z' is the zero group). Thus z' is an isomorphism from A to 0. (Indeed, similar reasoning shows that z is, too.)

For the second claim, let us label the maps

$$z : 0 \to A \qquad f : A \to B \qquad z' : B \to 0.$$

f is an injection by exactness at A. (The kernel of f is the image of z, which consists only of the element $0_A \in A$.) On the other hand, f is a surjection by exactness at B. (The image of f is the kernel of z', but z' is the zero map so its kernel is all of B.) This shows f is a group isomorphism. □

Example 8.1.6. Last class, when studying the homology of $X \coprod Y$, the Mayer-Vietoris sequence looks liked:

$$\begin{array}{ccccc}
 & & \ldots & \longrightarrow & H_{n+1}(X \coprod Y) \\
0 & \longrightarrow & H_n(X) \oplus H_n(Y) & \longrightarrow & H_n(X \coprod Y) \\
0 & \longrightarrow & H_{n-1}(X) \oplus H_{n-1}(Y) & \longrightarrow & H_{n-1}(X \coprod Y) \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
0 & \longrightarrow & H_1(X) \oplus H_1(Y) & \longrightarrow & H_1(X \coprod Y) \\
0 & \longrightarrow & H_0(X) \oplus H_0(Y) & \longrightarrow & H_0(X \coprod Y) \to 0
\end{array}$$

(The left column consisted of zero because $X \cap Y = \emptyset$ and all the homology groups of $\emptyset$ were declared to be zero.) By Proposition 8.1.5, all the maps $H_n(X) \oplus H_n(Y) \to H_n(X \coprod Y)$ must be group isomorphisms.

8.2 Fundamental facts

Exercise 8.2.1. edited as of February 22nd Let A be an abelian group.

Let Δ be the diagonal:

$$\Delta := \{(a, a) \mid a \in A\} \subset A \oplus A.$$

We also let $\overline{\Delta}$ be the antidiagonal:

$$\overline{\Delta} := \{(a, -a) \mid a \in A\} \subset A \oplus A.$$

(a) Show that the maps

$$(A \oplus A)/\Delta \to A, \qquad [(a,b)] \mapsto a - b, \qquad \text{and} \qquad [(a,b)] \mapsto b - a,$$

are both isomorphisms.

(b) Show that the maps

$$A \to (A \oplus A)/\Delta \qquad a \mapsto [(a,0)] \qquad \text{and} \qquad a \mapsto [(0,a)]$$

are both isomorphisms.

(c) Show that the maps

$$(A \oplus A)/\overline{\Delta} \to A, \qquad [(a,b)] \mapsto a + b, \qquad \text{and} \qquad [(a,b)] \mapsto -a - b,$$

are both isomorphisms.

(d) Show that the maps

$$A \to (A \oplus A)/\overline{\Delta} \qquad a \mapsto [(a,0)] \qquad \text{and} \qquad a \mapsto [(0,a)]$$

are both isomorphisms.

8.3 Path-connectedness and having the zeroth homology of a point

This section is new as of February 22

Recall the following notion from point-set topology:

Definition 8.3.1. A space X is called *path-connected* if X is non-empty[1], and for every $x_0, x_1 \in X$, there exists a continuous function $\gamma : [0,1] \to X$ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

[1] This is a somewhat annoying convention; it is similar to saying that 1 is not a prime number. In more advanced algebraic topology, the reason is that we are used to the fact that $\pi_0(X)$ – whatever π_0 is – consists of a single point if and only if X is path-connected. This necessitates that X is non-empty.

Many topologists have the following intuition: If a space X is path-connected, then $H_0(X;A) \cong A$. In other words, if zeroth homology of X is not just A, then X is not path-connected.

A sign of adulthood in algebra is whether one keeps track of isomorphisms, and not just the fact that groups are isomorphic. So here is a more sophisticated version of the above intuition:

Philosophy: If a space X is path-connected, then any map $pt \to X$ should induce an isomorphism on H_0.

Remark 8.3.2. If X is path-connected, any two maps $pt \to X$ are homotopic. So the induced map $H_0(pt;A) \to H_0(X;A)$ is well-defined regardless of the choice of $pt \to X$.

Let me first blow your mind with the fact that there are many kinds of homology for topological spaces, even when the coefficient group A is specified. All of them satisfy the axioms we have articulated so far, but these axioms do not uniquely determine homology. (They only determine homology on a collection of topological spaces called *CW complexes*, which we will see later in the course.)

And, it turns out that the above Philosophy is correct for the most common kind of homology and spaces that people study, but can sometimes be incorrect.

Today, we will see that many of our favorite spaces do satisfy this philosophy.

Definition 8.3.3. (This is a term used only in this course, and for the purposes of this reading. It is not used by other mathematicians.) Let's say that a space X *has the zeroth homology of a point* if X is path-connected, and if the[2] map

$$H_0(pt;A) \to H_0(X;A)$$

induced by a continuous function $pt \to X$ is an isomorphism for any A.

Remark 8.3.4. The term "has the zeroth homology" of a point should more accurately be stated "canonically has the zeroth homology" of a point. This is because we do not remember some abstract fact about $H_0(X;A)$ being isomorphic to $H_0(pt;A)$ – instead, we remember the actual isomorphism.

Let's see some examples.

[2]See Remark 8.3.2

Proposition 8.3.5. If X is homotopy equivalent to a point, then X has the zeroth homology of a point.

Proof. First let us show X is path connected. Because X is homotopy equivalent to a point, we know that there exist continuous functions $a : pt \to X$ and $X \to pt$ (this latter map is unique!) and a homotopy

$$H : X \times [0,1] \to X$$

such that $H(x,0) = x$ and $H(x,1) = a$.

In particular, $H(x,-) : [0,1] \to X$ defines a continuous path from x to a.

So choose two points $x_0, x_1 \in X$. Then $H(x_0,-)$ is a path γ_0 from x_0 to a, and $H(x_1,-)$ is a path from x_1 to a. By reversing time – i.e., with the map $[0,1] \to [0,1]$ taking $t \mapsto 1-t$ – the latter path becomes a path γ_1 from a to x_1.

By concatenating the two paths we obtain a path

$$\gamma : [0,1] \to X, \qquad t \mapsto \begin{cases} \gamma_0(2t) & t \in [0,1/2] \\ \gamma_1(2t-1) & t \in [1/2,1] \end{cases}$$

from x_0 to x_1. So X is path-connected.

Considering the same function $a : pt \to X$ as above, we know by homotopy invariance of homology that a_* is an isomorphism on all homology groups, and in particular, on H_0. We have proven that X has the zeroth homology of a point. □

Example 8.3.6. Choose an integer $n \geq 0$. We know that $\mathbb{R}^n$ is homotopy equivalent to a point. Thus, $\mathbb{R}^n$ has the zeroth homology of a point by Proposition 8.3.5.

Is this really a new example? Only in the attention we pay to it.

We already knew that if X is homotopy equivalent to a point, then all homology groups of X are isomorphic to those of a point. Proposition 8.3.5 is paying attention to the fact that X is in fact path-connected, and thus that any map from a point induces the desired isomorphism on homology.

Our goal is now to prove that many spaces have the zeroth homology of a point. The following will be useful to know:

Proposition 8.3.7. Suppose that X can be written as a union of two subsets U and V such that (i) $U \cup V = X$, (ii) $U \cap V$ is non-empty, and (iii) U and V are each path-connected. That X is path-connected.

Proof. Given two points $x_0, x_1 \in X$, if they are both contained in U it is obvious there is a continuous path between them (because U is assumed path-connected). Likewise if x_0, x_1 are both contained in V.

So assume without loss of generality that x_0 is contained in U, and x_1 is contained in $X \setminus U$ (which is a subset of V by (i)). Choose $y \in U \cap V$ (which we may, thanks to (ii)). Because U is path-connected, there exists a continuous path from x_0 to y. Because V is path-connected, there exists a continuous path from y to x_1. Concatenating these two paths, we see a continuous path from x_0 to x_1. This proves X is path-connected. □

8.3.1 Warm-up case

Proposition 8.3.8. Suppose that X can be written as a union of two open sets U and V such that U, V, and $U \cap V$ all have the zeroth homology of a point (Definition 8.3.3).

Then X has the zeroth homology of a point.

Remark 8.3.9. Proposition 8.3.8 is very useful. So is its proof. In fact, you may also want to remember the conclusion of Remark 8.3.11 below: It identifies the j_0 map in Mayer-Vietoris with the diagonal map.

Note we already know that X is path-connected by Proposition 8.3.7, so we must only understand the map on H_0 induced by an(y) inclusion of a point.

The 0th row of the Mayer-Vietoris sequence is:

$$H_0(U \cap V) \xrightarrow{j_0} H_0(U) \oplus H_0(V) \xrightarrow{i_0} H_0(X)$$

Goal: I want to very carefully understand the map j_0.

Notation 8.3.10. So choose some element of $U \cap V$; that is, choose a function

$$a : pt \to U \cap V.$$

Note that a is unique up to homotopy because we assumed $U \cap V$ is path-connected in this warm-up.

As usual, we have the two inclusions

$$j_U : U \cap V \to U, \qquad j_V : U \cap V \to V$$

in the set-up of Mayer-Vietoris. Let us define

$$a_U = j_U \circ a, \qquad a_V = j_V \circ a. \tag{8.3.1.1}$$

It is a good exercise to make sure you understand what the maps in (8.3.1.1) are. Each simply picks out a point of U and of V, respectively.

That's the topology.
Onto the algebra.

Remark 8.3.11 (Change of basis)**.** Because we assumed that U and V and $U \cap V$ have the zeroth homology of a point (Definition 8.3.3), the maps a_*, $(a_U)_*$ and $(a_V)_*$ are all isomorphisms.

Changing basis using these isomorphisms, we can transfer the information of j_0 to a map between the zeroth homology of points. Concretely, consider the following commutative diagram of abelian groups:

$$\begin{array}{ccc} H_0(U \cap V) & \xrightarrow{((j_U)_*,(j_V)_*)} & H_0(U) \oplus H_0(V) \\ \uparrow a_* & & \downarrow (a_U)_*^{-1} \oplus (a_V)_*^{-1} \\ H_0(pt) & \longrightarrow & H_0(pt) \oplus H_0(pt). \end{array}$$

The bottom horizontal arrow is computed as the composition of the other arrows. By matrix composition, we find that the bottom horizontal arrow equals

$$\begin{pmatrix} (a_U)_*^{-1} & 0 \\ 0 & (a_V)_*^{-1} \end{pmatrix} \begin{pmatrix} (a_U)_* \\ (a_V)_* \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{H_0(pt)} \\ \mathrm{id}_{H_0(pt)} \end{pmatrix}.$$

In other words, after changing basis to $H_0(pt; A)$, we see that the j_0 map in Mayer-Vietoris (given our particular hypotheses on U and V) is the diagonal embedding of $H_0(pt; A)$ into $H_0(pt; A) \oplus H_0(pt; A)$ – i.e., the map sends

$$x \mapsto (x, x).$$

We are almost done.
As usual in the zeroth row of the Mayer-Vietoris sequence

$$H_0(U \cap V) \xrightarrow{((j_U)_*,(j_V)_*)} H_0(U) \oplus H_0(V) \xrightarrow{(i_U)_* - (i_V)_*} H_0(X) \longrightarrow 0 \tag{8.3.1.2}$$

we see that $i_0 = (i_U)_* - (i_V)_*$ is a surjection by exactness at $H_0(X)$. By the first isomorphism theorem, the map induced by i_0

$$(H_0(U) \oplus H_0(V)) \,/\, \mathrm{im}(j_0) \to H_0(X), \qquad [(y, y')] \mapsto (i_U)_*(y) - (i_V)_*(y').$$

is an isomorphism. By Remark 8.3.11, we know there is an isomorphism

$$(H_0(pt) \oplus H_0(pt)) \,/ \Delta \to (H_0(U) \oplus H_0(V)) \,/\, \mathrm{im}(j_0), \qquad [(x, x')] \mapsto [((a_U)_*(x), (a_V)_*(x'))].$$

Composing the above two isomorphisms, we conclude that the map

$$(H_0(pt) \oplus H_0(pt)) \,/ \Delta \to H_0(X) \qquad [(x, x')] \mapsto (i_U)_*(a_U)_*(x) - (i_V)_*(a_V)_*(x') \tag{8.3.1.3}$$

is an isomorphism. On the other hand, for any abelian group K, the map

$$K \to (K \oplus K)/\Delta, \qquad x \mapsto [(x, 0)] \tag{8.3.1.4}$$

is an isomorphism (Exercise 8.2.1). Thus the composition of (8.3.1.3) and (8.3.1.4) for $K = H_0(pt)$ is an isomorphism, and can be computed as

$$H_0(pt) \to H_0(X), \qquad x \mapsto (i_U)_*(a_U)_*(x).$$

But $(a_U)_*$ was defined as a pushforward map, so this isomorphism can be rewritten as

$$\begin{aligned} x &\mapsto (i_U)_*(a_U)_*(x) \\ &= (i_U)_*(j_U \circ a)_*(x) \\ &= (i_U \circ j_U \circ a)_*(x). \end{aligned}$$

In other words, this isomorphism from $H_0(pt)$ to $H_0(X)$ is the induced map on homology of a continuous map $pt \to X$ given by $i_U \circ j_U \circ a$.

Because X is path-connected, this means that any map $pt \to X$ induces an isomorphism on H_0. This proves that X has the zeroth homology of a point, as desired.

8.3.2 When $U \cap V$ has two components

The following situation arises when we cover $X = S^1$ by two large open intervals. See Figure 8.4.1.

Proposition 8.3.12. Suppose that X can be written as a union of two open sets U and V such that U and V have the zeroth homology of a point, and $U \cap V$ is non-empty. Further assume that $U \cap V$ is a disjoint union of two spaces each having the zeroth homology of a point (Definition 8.3.3).

Then X has the zeroth homology of a point.

Remark 8.3.13. Compare Proposition 8.3.8 to Proposition 8.3.12. You clearly want to generalize to when $U \cap V$ has many components. This is possible, and is left for you as Exercise 8.6.7.

We only need the case of up to two connected components in $U \cap V$ – Proposition 8.3.12 – to get our feet off the ground (namely, to compute the homology of a circle).

Remark 8.3.14. As before, it will be useful to study this proof, especially Remark 8.3.17. Knowing an explicit expression for the map j_0 in the Mayer-Vietoris sequence (up to change of basis) will pay dividends.

Notation 8.3.15. We have assumed that $U \cap V$ is a disjoint union of two spaces with the zeroth homology of a point. Let's accordingly write

$$U \cap V = W_1 \coprod W_2$$

where each W_i has the homology of a point. Choose a point in each W_i:

$$a_1 : pt \to W_1, \qquad a_2 : pt \to W_2,$$

Each of these a_i induces an isomorphism $(a_i)_* : H_0(pt) \to H_0(W_i)$ by the assumption that W_i has the zeroth homology of a point.

We have inclusions

$$k_1 : W_1 \to U \cap V, \qquad k_2 : W_2 \to U \cap V,$$

and

$$j_U : U \cap V \to U, \qquad j_V : U \cap V \to V.$$

We finally let

$$a_{1,U} = j_U \circ k_1 \circ a_1, \qquad a_{2,V} = j_V \circ k_2 \circ a_2.$$

Remark 8.3.16 (Changing basis, again)**.** Let us again take the abstract map j_0 and re-write it in a basis we understand – i.e., as a map between (direct sums of) the zeroth homology of a point. We can make a commuting diagram

$$
\begin{array}{ccc}
H_0(U \cap V) & \xrightarrow{j_0} & H_0(U) \oplus H_0(V) \\
\uparrow {\scriptstyle ((k_1)_* - (k_2)_*)}\ \cong & & \\
H_0(W_1) \oplus H_0(W_2) & & \cong \downarrow {\scriptstyle ((a_{1,U})_*^{-1}, (a_{2,V})_*^{-1})} \\
\uparrow {\scriptstyle (a_1)_* \oplus (a_2)_*}\ \cong & & \\
H_0(pt) \oplus H_0(pt) & \longrightarrow & H_0(pt) \oplus H_0(pt)
\end{array}
\tag{8.3.2.1}
$$

by declaring the bottom horizontal arrow to be the composition of the other arrows.

Remark 8.3.17. Let us now compute the bottom horizontal arrow. We perform the following matrix multiplication:

$$
\begin{pmatrix} (a_{1,U})_*^{-1} & 0 \\ 0 & (a_{2,V})_*^{-1} \end{pmatrix}
\begin{pmatrix} (j_U)_* \\ (j_V)_* \end{pmatrix}
\begin{pmatrix} (k_1)_* & -(k_2)_* \end{pmatrix}
\begin{pmatrix} (a_1)_* & 0 \\ 0 & (a_2)_* \end{pmatrix}
$$
$$
= \begin{pmatrix} (a_{1,U})_*^{-1} & 0 \\ 0 & (a_{2,V})_*^{-1} \end{pmatrix}
\begin{pmatrix} (j_U)_* \\ (j_V)_* \end{pmatrix}
\begin{pmatrix} (k_1 \circ a_1)_* & -(k_2 \circ a_2)_* \end{pmatrix}
$$
$$
= \begin{pmatrix} (a_{1,U})_*^{-1} & 0 \\ 0 & (a_{2,V})_*^{-1} \end{pmatrix}
\begin{pmatrix} (j_U \circ k_1 \circ a_1)_* & -(j_U \circ k_2 \circ a_2)_* \\ (j_V \circ k_1 \circ a_1)_* & -(j_V \circ k_2 \circ a_2)_* \end{pmatrix}.
$$

The matrix on the right looks like a doozy, but let us make two observations. First, $j_U \circ k_2 \circ a_2$ – which picks out a point of U – is a continuous map that is homotopic to $j_U \circ k_1 \circ a_1$. This is because U is assumed path-connected. Likewise, we have

$$j_V \circ k_1 \circ a_1 \sim j_V \circ k_2 \circ a_2$$

because V is path-connected. We conclude:

$$(j_V \circ k_1 \circ a_1)_* = (j_V \circ k_2 \circ a_2)_*, \qquad (j_U \circ k_1 \circ a_1)_* = (j_U \circ k_2 \circ a_2)_*.$$

Noting the definitions of $a_{1,U}$ and $a_{1,V}$, we may continue our matrix multiplication computation as follows:

$$
= \begin{pmatrix} (a_{1,U})_*^{-1} & 0 \\ 0 & (a_{2,V})_*^{-1} \end{pmatrix}
\begin{pmatrix} (a_{1,U})_* & -(a_{1,U})_* \\ (a_{2,V})_* & -(a_{2,V})_* \end{pmatrix}
$$
$$
= \begin{pmatrix} \mathrm{id}_{H_0(pt)} & -\mathrm{id}_{H_0(pt)} \\ \mathrm{id}_{H_0(pt)} & -\mathrm{id}_{H_0(pt)} \end{pmatrix}.
$$

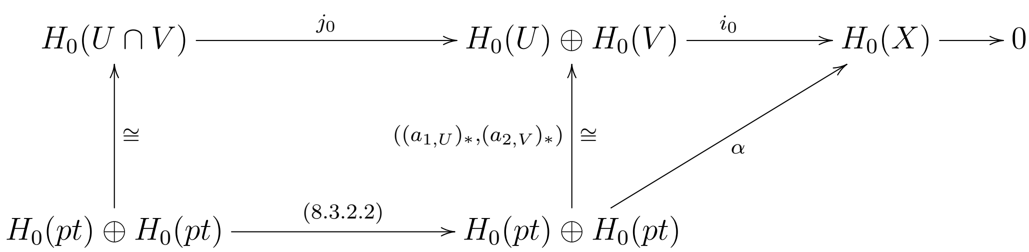
Explicitly, we have found that the bottom horizontal arrow in (8.3.2.1) is the homomorphism sending

$$(x, y) \mapsto (x - y, x - y). \tag{8.3.2.2}$$

Example 8.3.18. If $A = \mathbb{Z}$ and we fix an isomorphism $H_0(pt; \mathbb{Z}) \cong \mathbb{Z}$, then the matrix above can be rewritten as a matrix of integers as follows:

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The rest of the proof of Proposition 8.3.12. The top row in the diagram below is the 0th row of the Mayer-Vietoris sequence:



The upward-rightward arrow α is define to be the composition of the other two maps in the triangle. By exactness, we know that i_0 is a surjection. So by the first isomorphism theorem, we know that $H_0(X)$ is isomorphic to $H_0(U) \oplus H_0(V)/\ker(i_0)$. By exactness, $\ker(i_0) = \operatorname{im}(j_0)$. Because the vertical arrows are isomorphisms, we conclude that the upward-rightward map α above induces an isomorphism

$$(H_0(pt) \oplus H_0(pt))/\operatorname{im}((8.3.2.2)) \to H_0(X).$$

Concretely (by definition of α and i_0) this map acts by

$$[(x, y)] \mapsto (i_U \circ a_{1,U})_*(x) - (i_V \circ a_{2,V})_*(y).$$

Let us now consider the composition

$$H_0(pt) \to (H_0(pt) \oplus H_0(pt))/\operatorname{im}((8.3.2.2)) \to H_0(X)$$

where the first arrow sends an element x to $[(x, 0)]$. Because the image of (8.3.2.2) is the diagonal, we know the first arrow is an isomorphism (Exercise 8.2.1). Because we just saw that the second arrow is an isomorphism, the above composition is an isomorphism. Moreover, parsing the formulas, the composition is precisely

$$x \mapsto (i_U \circ a_{1,U})_*(x).$$

That is, it is the map on 0th homology induced by the continuous function $i_U \circ a_{1,U} : pt \to X$.

This shows that the map from a(ny) point to X induces an isomorphism on H_0, and proves the proposition. □

8.4 Homology of the circle

Re-written as of February 22

Let $U = \{(x_0, x_1) \in S^1 \mid x_1 < 1\}$ and $V = \{(x_0, x_1) \in S^1 \mid x_1 > -1\}$. Note that $U \cap V$ consists of two disjoint open intervals of S^1. See Figure 8.4.1 for some visualizations.

Note that $U \cong V \cong \mathbb{R}$, and $U \cap V \cong \mathbb{R} \coprod \mathbb{R}$. Because $\mathbb{R}$ is homotopy equivalent to a point, everything in sight has the zeroth homology of a point (Example 8.3.6). So we may apply Proposition 8.3.12 to conclude that $H_0(S^1) \cong H_0(pt)$. By 5.4.1, we thus see

$$H_0(S^1) \cong A.$$

In Row 0 of the Mayer-Vietoris sequence, we have the map

$$H_0(U \cap V) \xrightarrow{j_0} H_0(U) \oplus H_0(V). \tag{8.4.0.1}$$

By Remark 8.3.17, we know j_0 (after a change of basis) is the map (8.3.2.2), which we reproduce here for the reader's convenience:

$$H_0(pt) \oplus H_0(pt) \to H_0(pt) \oplus H_0(pt), \qquad (x, y) \mapsto (x - y, x - y).$$

So we have

$$\ker((8.3.2.2)) = \Delta = \{(x, y) \mid x = y\}.$$

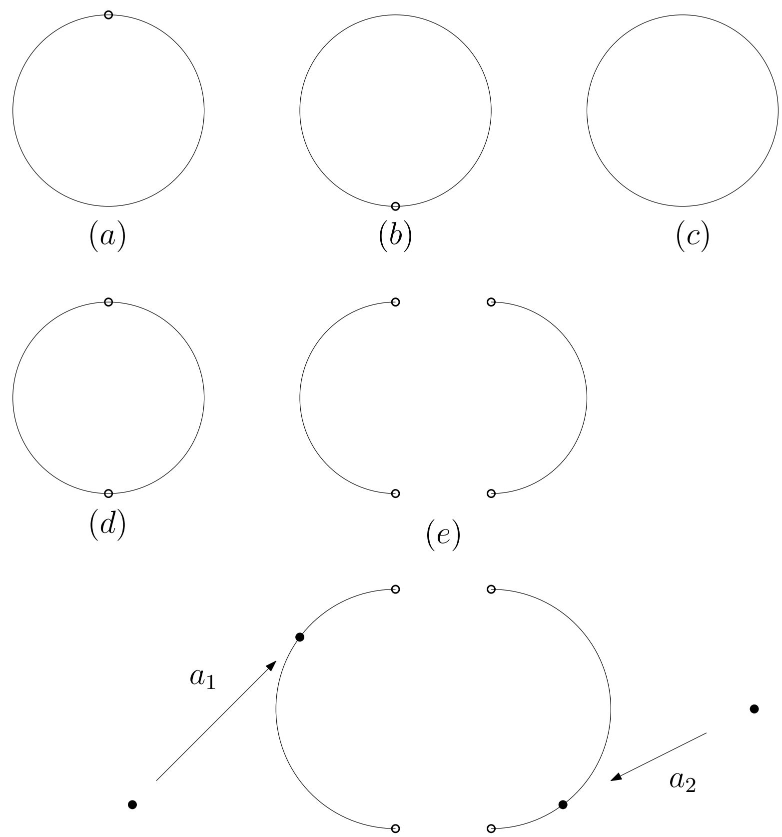


Figure 8.4.1. (a) depicts the open subset U. (b) is the open subset V. (c) is the circle S^1. (d) is the intersection $U \cap V$. (e) is a redrawing of $U \cap V$ to emphasize that $U \cap V$ is a disjoint union of two open intervals. In the main text, W_1 is the name given to the lefthand component, while W_2 is the righthand component.
The bottom of this figure depicts two continuous functions from two different points: a_1 is a map from a point picking out one connected component of $U \cap V$, while a_2 is a function from a point picking out the other connected component.

With this preparation, let us now study the Mayer-Vietoris sequence:

$$\begin{array}{ccccc}
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
H_2(U \cap V) & \longrightarrow & H_2(U) \oplus H_2(V) & \longrightarrow & H_2(S^1) \\
H_1(U \cap V) & \longrightarrow & H_1(U) \oplus H_1(V) & \longrightarrow & H_1(S^1) \\
H_0(U \cap V) & \longrightarrow & H_0(U) \oplus H_0(V) & \longrightarrow & H_0(S^1) \to 0
\end{array}$$

We note that $U \cap V \simeq pt \coprod pt$ and $U \simeq pt, V \simeq pt$, so[3] all the homology groups of the middle and lefthand columns vanish for $n \geq 1$:

$$\begin{array}{ccccc}
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & H_2(S^1) \\
0 & \longrightarrow & 0 & \longrightarrow & H_1(S^1) \\
H_0(U \cap V) & \xrightarrow[j_0]{} & H_0(U) \oplus H_0(V) & \longrightarrow & H_0(S^1) \to 0
\end{array}$$

We thus[4] conclude that

$$H_n(S^1) \cong 0 \text{ for all } n \geq 2.$$

We also know by Proposition 8.3.12

$$H_0(S^1) \cong H_0(pt) \cong A.$$

. So it remains to compute $H_1(S^1)$. By exactness, we know that the connecting homomorphism $H_1(S^1) \to H_0(U \cap V)$ is an injection[5] So $H_1(S^1)$ is

[3] This is because we know $H_n(pt)$ and we know $H_n(pt \coprod pt) \simeq H_n(pt) \oplus H_n(pt)$.

[4] See Remark 7.4.1.

[5] The kernel of the connecting map, by exactness, equals the image of the previous map. And the previous map is the zero map, so has trivial image.

isomorphic to the image of the connecting homomorphism. On the other hand, by exactness, this image is equal to the kernel of j_0, which we have already computed to be a group isomorphic to the diagonal of $H_0(pt) \oplus H_0(pt)$. But of course the diagonal is isomorphic to $H_0(pt)$, so we conclude that $H_1(S^1) \cong H_0(pt) \cong A$.

8.5 Trees

First, recall that a *tree* is a non-empty, connected graph with no cycles. For our purposes, a tree is also a topological space – interiors of edges are topologized to be homeomorphic to $\mathbb{R}$, for example.

Theorem 8.5.1. Any tree is homotopy equivalent to a point.

Proof sketch. The proof depends very much on your definition of "tree." Some definitions make this proof a major theorem, while others make it rather trivial.

Regardless of your definition of tree, the following fact is true: Given any two points u, v of the tree T , there exists a unique (up to parametrization) continuous path $\gamma_{u,v} : [0,1] \to T$ with $\gamma(0) = u$ and $\gamma(1) = v$. Fix u. The $\gamma_{u,v}$ can be chosen to depend continuously on the choice of v, so we have a continuous function

$$T \times [0,1] \to T, \qquad (v,t) \mapsto \gamma_{u,v}(t)$$

which at $t = 0$ is the constant map with image u, and at $t = 1$ is the identity. This homotopy can be used to show that the inclusion $pt \to T$, sending the point to u, is a homotopy equivalence. □

Remark 8.5.2. This theorem is even more subtle than it looks. For example, I never postulated that T is a finite tree. We don't want to spend all day topologizing a tree with infinitely many vertices and edges in the "correct" way, and we also won't deal with infinitely large trees in our course, so we will ignore this point.

Exercise 8.5.3. Let T be a tree. Compute the homology groups of T. (Use Theorem 8.5.1.)

8.6 Exercises

Exercise 8.6.1. Show that for $A = \mathbb{Z}$ or $A = \mathbb{F}_2$,, the homology groups of the figure 8 graph are given as follows:

$$H_n(8; A) \cong \begin{cases} A & n = 0 \\ A \oplus A & n = 1 \\ 0 & n \geq 2. \end{cases}$$

As a hint, you will want to use Mayer-Vietoris, Exercise 8.5.3, Exercise 8.2.1, Exercise 8.6.5, and the first isomorphism theorem.

(This exercise is true regardless of choice of A, but we take $A = \mathbb{Z}$ and $A = \mathbb{F}_2$ for concreteness.)

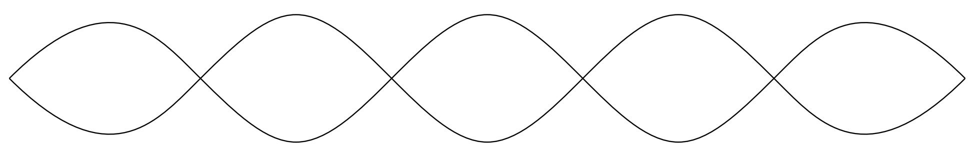


Figure 8.6.2. A chain of five circles.

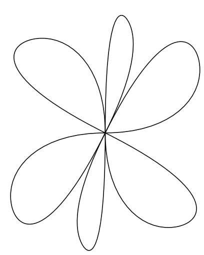


Figure 8.6.3. A bouquet of six circles.

Exercise 8.6.4. Let X be a chain of k circles, or a bouquet of k circles. (See Figures 8.6.2 and 8.6.3 for what I mean.) Show that

$$H_n(X; A) \cong \begin{cases} A & n = 0 \\ A^{\oplus k} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Exercise 8.6.5. Suppose X is a topological space with $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. Prove that for any $x \in X$, the continuous function

$$f : pt \to X$$

sending the point to x induces an isomorphism on $H_0(X;\mathbb{Z})$.

Prove the same with $\mathbb{F}_2$ coefficients.

Hint: The constant map $p : X \to pt$ is continuous, and of course $p \circ f = \mathrm{id}_{pt}$. The induced map f_* is thus an injection on H_0, while p_* is a surjection on H_0. On the other hand $X \to pt \to X$ squares to itself; so $(fp)_*$ is a projection operator $A \to A$ with image isomorphic to A. The only such projection operators for $A = \mathbb{Z}$ and $A = \mathbb{F}_2$ are the identity.

Remark 8.6.6. When one gives a concrete model for homology, Exercise 8.6.5 is much simpler to establish. However, the trickiness of the above exercise is a hint that the axiomatics leave room for much more general kinds of homology theories. Indeed, there are extraordinary homology theories out there that do not simply arise from coefficient groups A – these are sometimes called spectra, or stable homotopy types.

Exercise 8.6.7. Prove the following generalization of Proposition 8.3.12:

Suppose that X can be written as a union of two open sets U and V such that U and V have the zeroth homology of a point, and $U \cap V$ is non-empty. Further assume that $U \cap V$ is a finite disjoint union of spaces each having the zeroth homology of a point (Definition 8.3.3).

Then X has the zeroth homology of a point.