Class Notes 2/12/24

curvy arrow implies injection

Double arrow implies surjection

Defn: A sequence of Abelian grp homomorphisms

$$\cdots \to A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} A_{i-2} \to \cdots$$

is called *exact* if $\forall\, i$, $\ker(f_i) = \text{Image}(f_{i+1})$

- An exact sequence $0 \hookrightarrow A \twoheadrightarrow B \to C \to 0$ is called a *short* exact sequence

Here, the arrow from A to B should be the injection arrow (hook butt) and the arrow from B to C should be the surjection arrow (double head).

- Any other exact sequence is called *long*. (typically longer than three terms)

Questions

Recall 1st ISOMORPHISM THEOREM:

Given $f: G \to H$, $G/\ker(f) \cong \text{image}(f)$

Applying it to mapping above, we get:

$$B/\underbrace{\ker(g)}_{\text{Im}(f)} \cong \underbrace{\text{image}(f)}_{C}$$

$$\longrightarrow \quad B/A \cong C$$

Prop: If: $0 \xrightarrow{f_{i+1}} A \xrightarrow{f_i} B \xrightarrow{f_{i-1}} 0$

then f_i is an isomorphism.

Proof:

By exactness at A, $\ker(f) = \text{Image}(0 \to A)$
$= \{0_A\}$

- If the kernel is trivial, f is an injection. ✓

- By exactness at B,

$B = \ker(B \to 0) = \text{Image}(f)$

Then f is a surjection. ✓

Hence, f is an isomorphism.

Questions

$f : X \to Y$

$f_* : H_n(X; A) \to H_n(Y; A)$

lower star means induced map

$i_V : V \to X$

$i_U : U \to X$

$U \cap V \xrightarrow{j_U} U$

$U \cap V \xrightarrow{j_V} V$

We can do this bc these are homomorphisms

If $X = U \cap V$ where U, V are open:

$$H_2(U \cap V) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(X) \xrightarrow{\delta}$$
$$H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(X) \xrightarrow{\delta}$$
$$H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(X) \longrightarrow 0$$

Thm: (last time), there exist these blue arrows δ, s.t. the sequence is exact.

Example: $X = U \amalg V$ as a space

$$H_2(\emptyset) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(X) \xrightarrow{\delta}$$
$$H_1(\emptyset) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(X) \xrightarrow{\delta}$$
$$H_0(\emptyset) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(X) \longrightarrow 0$$

The homology of the empty set is 0 for all n, Hence:

$$\overset{0}{\cancel{H_2(\emptyset)}} \longrightarrow H_2(U) \oplus H_2(V) \xrightarrow{\cong} H_2(X) \xrightarrow{\delta}$$
$$\overset{0}{\cancel{H_1(\emptyset)}} \longrightarrow H_1(U) \oplus H_1(V) \xrightarrow{\cong} H_1(X) \xrightarrow{\delta}$$
$$\overset{0}{\cancel{H_0(\emptyset)}} \longrightarrow H_0(U) \oplus H_0(V) \xrightarrow{\cong} H_0(X) \longrightarrow 0$$

So, These are isomorphisms by Prop. above.

So $H_n(X) \cong H_n(U) \oplus H_n(V)$

So this would work for any disjoint sets (i.e. disjoint points)

Question: What does $A \oplus A / \Delta$ mean?

Fix an Abelian group

$$A \oplus A = \{ (a_1, a_2) \mid a_1, a_2 \in A \}$$

where $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

Notation:

$\Delta := \{ \underbrace{(a, a)}_{\text{same values}} \mid a \in A \}$

diagonal

$= \{ (a_1, a_2) \mid a_1 = a_2 \} \subset A \oplus A$

Example (of $A \oplus A / \Delta$) for $A = \mathbb{Z}/4\mathbb{Z}$

$$[(a_1, a_2)] \in A \oplus A / \Delta$$

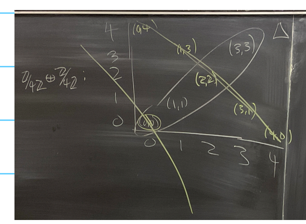
e.g. $[(0,0)]$ $[(1,1)]$ $[(2,3)]$ $[(1,2)]$ $[(1,3)]$ $[(0,2)]$ ...

$[(a_1, a_2)] = [(b_1, b_2)] \iff \exists\, c$ s.t. $(a_1, a_2) + (c, c) = (b_1, b_2)$

So These are the same "diagonally" bc you could add $(1,1)$ to $(1,2)$ and get $(2,3)$

On the reading, we need to show that

$$[(a_1, a_2)] \in \boxed{A \oplus A / \Delta \xrightarrow{\cong} A}$$



Question: Why is $\bar{\Delta}$ called an antidiagonal?

$[(2,3)]$ and $[(1,2)]$ are not related antidiagonally

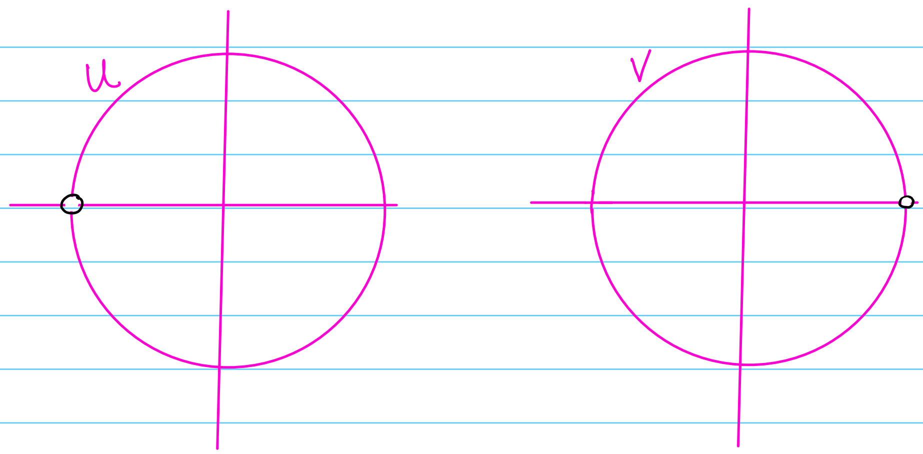
but $[(2,1)]$ and $[(1,2)]$ are related antidiagonally

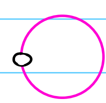
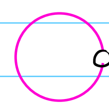
Today: let's compute the homology of a circle.
$H_*(S^1)$

We start the M.V. sequence by choosing an $\mathcal{U}$ & V

$$\mathcal{U} = \{ (x_1, x_2) \in S^1 \mid x_1 > -1 \}$$

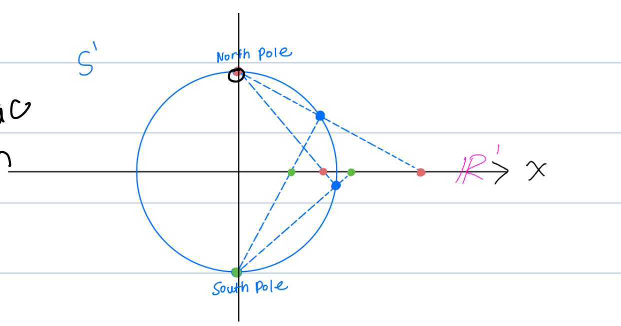
$$V = \{ (x_1 x_2) \in S^1 \mid x_1 < 1 \}$$

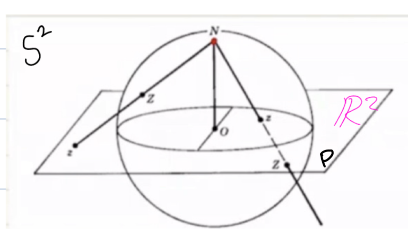


$\mathcal{U} =$  $\overset{\pi}{\cong} \mathbb{R}^1 \overset{\pi}{\cong}$  $= V$ we also know $\mathbb{R}^1 \cong *_{pt}$

spaghetti

stereographic projection





Hence, $S^{n-1} \setminus \{x\} \to P \cong \mathbb{R}^{n-1}$

$y \longmapsto \pi(y) = P \cap$ line through x, y

Thm. we can take for granted:
π is a homeomorphism

(If two spaces are homeomorphic, then their homology groups are isomorphic)

Now, onto the M.V. Sequence

$$
\begin{array}{ccccccc}
H_2(U\cap V) & \longrightarrow & H_2(U)\oplus H_2(V) & \longrightarrow & H_2(X) & \xrightarrow{\delta} & \\
H_1(U\cap V) & \longrightarrow & H_1(U)\oplus H_1(V) & \longrightarrow & H_1(X) & \xrightarrow{\delta} & \\
H_0(U\cap V) & \longrightarrow & H_0(U)\oplus H_0(V) & \longrightarrow & H_0(X) & \longrightarrow & 0
\end{array}
$$

Do we know the homology groups of U?
Well we know the homology groups of $\mathbb{R}$

Recall:
$H_n(pt) = \begin{cases} A & \text{if } n=0 \\ 0 & \text{elsewhere} \end{cases}$

$$
\begin{array}{ccccccc}
H_2(U\cap V) & \longrightarrow & \overset{0}{\cancel{H_2(U)}}\oplus H_2(V) & \longrightarrow & H_2(X) & \xrightarrow{\delta} & \\
H_1(U\cap V) & \longrightarrow & \overset{0}{\cancel{H_1(U)}}\oplus H_1(V) & \longrightarrow & H_1(X) & \xrightarrow{\delta} & \\
H_0(U\cap V) & \longrightarrow & \underset{A}{\cancel{H_0(U)}}\oplus H_0(V) & \longrightarrow & H_0(X) & \longrightarrow & 0
\end{array}
$$

bc $H_n(\mathbb{R}) \cong H_n(pt) = A$

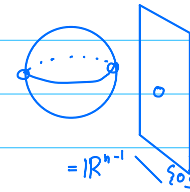
by same argument

$$
\begin{array}{ccccccc}
H_2(U\cap V) & \longrightarrow & 0 \oplus \overset{0}{\cancel{H_2(V)}} & \longrightarrow & H_2(S^1) & \xrightarrow{\delta} & \\
H_1(U\cap V) & \longrightarrow & 0 \oplus \overset{0}{\cancel{H_1(V)}} & \longrightarrow & H_1(S^1) & \xrightarrow{\delta} & \\
H_0(U\cap V) & \longrightarrow & A \oplus \overset{A}{\cancel{H_0(V)}} & \longrightarrow & H_0(S^1) & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccc}
H_2(U\cap V) & \longrightarrow & 0 & \longrightarrow & H_2(S^1) & \xrightarrow{\delta} & \\
H_1(U\cap V) & \longrightarrow & 0 & \longrightarrow & H_1(S^1) & \xrightarrow{\delta} & \\
H_0(U\cap V) & \longrightarrow & A \oplus \overset{A}{\cancel{H_0(V)}} & \longrightarrow & H_0(S^1) & \longrightarrow & 0
\end{array}
$$

Hence, since $\ker(f) = \operatorname{Im}(f_{i+1})$, so

$$\begin{tikzcd}
H_2(U\cap V) \arrow[r] & 0 \arrow[r] & H_2(S^1) \arrow[dll, "\delta"'] \\
H_1(U\cap V) \arrow[r] & 0 \arrow[r] & H_1(S^1) \arrow[dll, "\delta"'] \\
H_0(U\cap V) \arrow[r] & A \oplus A \arrow[r] & H_0(S^1) \arrow[r] & 0
\end{tikzcd}$$



Will need to use this & induction on HW

$U \cap V = $ ⊖ $\cong \mathbb{R} \amalg \mathbb{R}$

But $\mathbb{R} \amalg \mathbb{R} \cong pt \amalg pt$

So, now

$$\begin{tikzcd}
\cancel{H_2(U\cap V)}^{0} \arrow[r] & 0 \arrow[r] & H_2(S^1) \arrow[dll, "\delta"'] \\
\cancel{H_1(U\cap V)}^{0} \arrow[r] & 0 \arrow[r] & H_1(S^1) \arrow[dll, "\delta"'] \\
\underset{A\oplus A}{\cancel{H_0(U\cap V)}} \arrow[r] & A \oplus A \arrow[r] & H_0(S^1) \arrow[r] & 0
\end{tikzcd}$$

Cleaning this up:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "f"] & H_2(S^1) \arrow[dll, "\delta"'] \\
0 \arrow[r] & 0 \arrow[r] & H_1(S^1) \arrow[dll, "\delta"'] \\
A \oplus A \arrow[r] & A \oplus A \arrow[r] & H_0(S^1) \arrow[r] & 0
\end{tikzcd}$$

$\cong \Delta \cong \boxed{A}$ injection

surjection $\overset{so}{\cong} \boxed{A}$

So the image of f must equal 0.
And hence the kernel of δ must equal $\operatorname{Im}(f) = 0$

$$\Longrightarrow H_2(S^1) = 0$$

Whatever this map is, it'll look like $\begin{pmatrix} 1 & \pm 1 \\ 1 & \pm 1 \end{pmatrix}$
So this will either look like the diagonal or antidiagonal.
So $H_0(S^1) = A$