Reading 7

Mayer-Vietoris and long exact sequences

We are gathering tools to compute homology. We will take for granted the homology of a point:

$$H_n(pt; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.0.0.1}$$

Last time we saw that homotopy equivalences are powerful weapons. If two spaces are homotopy equivalent, their homology groups are isomorphic. This allowed us to compute the homology groups of Euclidean space of any dimension k:

$$H_n(\mathbb{R}^k; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Today we will see that homology also satisfies a "local-to-global" property: By computing homology on bits and pieces of a space, we have hope of computing the homology of the whole space.

Notation 7.0.1. It is common to suppress the coefficient group A from homology notation, meaning we use the shortened notation

$$H_n(X)$$

to mean $H_n(X; A)$. In other words, we will often leave A "implicit" in the notation.

7.1 Natural maps from a two-set cover

Suppose that U and V are two (not necessarily open) subsets of X. By the definition of subspace topology, inclusions of subspaces are continuous. So we have several continuous maps arising from U and V:

$$\begin{aligned} i_U &: U \to X, \\ i_V &: V \to X, \\ j_U &: U \cap V \to U, \\ j_V &: U \cap V \to V. \end{aligned}$$

So for all n, we have induced maps on homology

$$\begin{aligned} (i_U)_* &: H_n(U) \to H_n(X), \\ (i_V)_* &: H_n(V) \to H_n(X), \\ (j_U)_* &: H_n(U \cap V) \to H_n(U), \\ (j_V)_* &: H_n(U \cap V) \to H_n(V). \end{aligned}$$

For reasons that only become clear with experience, I want to consider the composition

$$H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(X) \tag{7.1.0.1}$$

where the first arrow is given by

$$H_n(U \cap V) \to H_n(U) \oplus H_n(V), \qquad x \mapsto ((j_U)_*(x), (j_V)_*(x))$$

and the second arrow is given by

$$H_n(U) \oplus H_n(V) \to H_n(X), \qquad (a, b) \mapsto (i_U)_*(a) - (i_V)_*(b). \tag{7.1.0.2}$$

Remark 7.1.1. Perhaps the most perplexing part of the above maps is the minus sign in (7.1.0.2). The idea is to make sure that the contribution from $U \cap V$ cancels inside X. Indeed, the composition $i_U \circ j_U$ is equal to the composition $i_V \circ j_V$, so the minus sign guarantees that the composition of the two arrows in (7.1.0.1) is zero.

7.2 Homology of the empty set

So far I've only "told" you the homology of one space: The point. From this and invariance under homotopy equivalence, we could compute the homology of Euclidean space.

Today let me just tell you one more fact:

Fact 7.2.1. The empty set has zero homology groups in every degree.

That is, for every A and every $n \geq 0$,

$$H_n(\emptyset; A) \cong 0. \tag{7.2.0.1}$$

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7.3 The Mayer-Vietoris sequence

Theorem 7.3.1. Fix a topological space X. Let U and V be open subsets of X such that $U \cup V = X$. Then for any coefficient group A, there exist diagonal maps as below:

$$\begin{array}{ccccc}
 & & \ldots & \longrightarrow & H_{n+1}(X) \\
 & \swarrow & & & \\
H_n(U \cap V) & \longrightarrow & H_n(U) \oplus H_n(V) & \longrightarrow & H_n(X) \\
 & \swarrow & & & \\
H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) & \longrightarrow & H_{n-1}(X) \\
 & \swarrow & & & \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
 & \swarrow & & & \\
H_1(U \cap V) & \longrightarrow & H_1(U) \oplus H_1(V) & \longrightarrow & H_1(X) \\
 & \swarrow & & & \\
H_0(U \cap V) & \longrightarrow & H_0(U) \oplus H_0(V) & \longrightarrow & H_0(X) \to 0
\end{array}$$

such that, when the horizontal arrows are as in (7.1.0.1), the kernel of each map is the image of the previous map.

The sequence in Theorem 7.3.1 is called the *Mayer-Vietoris sequence* associated to the choice of U and V.

The diagonal arrows, which have domain and codomain as follows:

$$H_n(X; A) \to H_{n-1}(U \cap V; A),$$

are often called the *connecting homomorphisms* or the *boundary homomorphisms*.

7.3.1 Details for parsing the Mayer-Vietoris sequence

Remark 7.3.2. In the sequence of groups above, we witness the homology groups of U, V, and $U \cap V$. Each of these $U, V, U \cap V$ is treated as a topological space (via the subspace topology inherited from X) so it makes sense to ask for their homology groups.

Remark 7.3.3. There is a version of Mayer-Vietoris where we may relax the hypothesis: Rather than U and V being open in X, we may ask that the interiors of U and of V cover X. We will not need this version in this class.

For whatever reason, sequences of group homomorphisms where the image of each map equals the kernel of the next arises – a lot – in math. So we give such a sequence of maps an adjective: *exact*.

Definition 7.3.4. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two group homomorphisms. We say this collection of homomorphisms is *exact at* B if $\ker g = \mathrm{im}(f)$. (That is, if the kernel of g is the image of f.)

More generally, given a sequence of group homomorphisms with the domain of each homomorphism matching the codomain of the previous homomorphism, we say the sequence is an *exact sequence* if the kernel of every map is the image of the previous map.

Example 7.3.5. The Mayer-Vietoris sequence is an *exact* sequence of maps.

7.4 Example: Disjoint union of two points

We begin with perhaps the simplest example.

Suppose that X can be written as a union of two open subsets that do not intersect. An example is when $X = pt \coprod pt \cong \{a, b\}$ is a disjoint union of two points. Then the subsets $\{a\}$ and $\{b\}$ are both open subsets of X.[1] Set

$$U = \{a\}, \qquad V = \{b\}.$$

Note that $U \cong V \cong pt$, so we know the homologies of these sets. Note also that $U \cap V = \emptyset$, so the Mayer-Vietoris sequence becomes the following:

$$\begin{array}{ccccc}
 & & \ldots & \longrightarrow & H_{n+1}(pt \coprod pt) \\
H_n(\emptyset) & \longrightarrow & H_n(pt) \oplus H_n(pt) & \longrightarrow & H_n(pt \coprod pt) \\
H_{n-1}(\emptyset) & \longrightarrow & H_{n-1}(pt) \oplus H_{n-1}(pt) & \longrightarrow & H_{n-1}(pt \coprod pt) \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
H_1(\emptyset) & \longrightarrow & H_1(pt) \oplus H_1(pt) & \longrightarrow & H_1(pt \coprod pt) \\
H_0(\emptyset) & \longrightarrow & H_0(pt) \oplus H_0(pt) & \longrightarrow & H_0(pt \coprod pt) \to 0
\end{array}$$

Now let us plug in what we know about the homology of the empty set (7.2.0.1) and the point (7.0.0.1), with say $A = \mathbb{F}_2$ coefficients:

[1] There exists a topology on $\{a, b\}$ for which this statement is not true! Indeed, the disjoint union forces X to have the *discrete* topology, where every subset is open.

$$\begin{array}{ccccc}
 & & \ldots & \longrightarrow & H_{n+1}(pt \coprod pt) \\
 & \swarrow & & & \\
0 & \longrightarrow & 0 & \longrightarrow & H_n(pt \coprod pt) \\
 & \swarrow & & & \\
0 & \longrightarrow & 0 & \longrightarrow & H_{n-1}(pt \coprod pt) \\
 & \swarrow & & & \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
 & \swarrow & & & \\
0 & \longrightarrow & 0 & \longrightarrow & H_1(pt \coprod pt) \\
 & \swarrow & & & \\
0 & \longrightarrow & \mathbb{F}_2\text{‘} \oplus \mathbb{F}_2 & \longrightarrow & H_0(pt \coprod pt) \to 0
\end{array}$$

Now we shall use the most important property of the Mayer-Vietoris sequence: It is exact. We don't even need to know exactly what the diagonal maps are in this example! Indeed, let use understand what it means for the following part of the sequence to be exact:

$$0 \xrightarrow{j_*} \mathbb{F}_2 \oplus \mathbb{F}_2 \xrightarrow{(i_U)_* - (i_V)_*} H_0(pt \coprod pt) \longrightarrow 0$$

where the last arrow is the boundary/connecting homomorphism. (We do not draw it diagonally, for sake of space.)

The sequence is exact at $H_0(pt \coprod pt)$, which means that the kernel of the last map is the image of the previous map. Well, the last map is the zero map! So all of $H_0(pt \coprod pt)$ is the kernel. By exactness, the image if $(i_U)_* - (i_V)_*$ must therefore be all of $H_0(pt \coprod pt)$. We conclude that the middle arrow must be a surjection.

The sequence is exact at $\mathbb{F}_2 \oplus \mathbb{F}_2$. This means the image of j_* is the kernel of the middle arrow. Well, j_* is the zero map! So its image is trivial. (Its image consists only of $0 \in \mathbb{F}_2 \oplus \mathbb{F}_2$.) This means the kernel of the middle arrow is trivial, which means that the middle arrow is an injection.

We conclude that the middle arrow is a bijection.

So, we have proven the following:

$$H_0(pt \coprod pt) \cong \mathbb{F}_2 \oplus \mathbb{F}_2.$$

How about for $n \geq 1$? Well, we have that the following portion of the Mayer-Vietoris sequence is exact:

$$0 \longrightarrow H_n(pt \coprod pt) \longrightarrow 0. \tag{7.4.0.1}$$

The image of the horizontal arrow is zero, of course; so (by exactness) the diagonal arrow has trivial kernel, meaning the diagonal arrow must be an injection. But if $H_n(pt \coprod pt)$ admits an injection to the zero group, $H_n(pt \coprod pt)$ must be (isomorphic to) the zero group.

Remark 7.4.1. The reasoning around (7.4.0.1) happens often. If you have an exact sequence

$$0 \to A \to 0$$

then A must be (isomorphic to) the zero group.

7.5 Example: Disjoint union of spaces

Exercise 7.5.1. (a) Suppose X and Y are topological spaces. Let $X \coprod Y$ be the disjoint union (as a set). We can endow $X \coprod Y$ with the *disjoint union topology*, where a subset $U \subset X \coprod Y$ is open if and only if both $U \cap X$ and $U \cap Y$ are open (in the original topologies of X and Y, respectively).

If you have time, prove that this is indeed a topology on $X \coprod Y$. Even if you do have time, it's okay to skip this exercise.

(b) For every n and A, show that there exists an isomorphism

$$H_n(X; A) \oplus H_n(Y; A) \cong H_n(X \coprod Y; A)$$

induced by the inclusion maps $X \to X \coprod Y$ and $Y \to X \coprod Y$.

(c) Suppose we have a finite collection of topological spaces $X_1, \ldots, X_k$ and let $X = X_1 \coprod \ldots \coprod X_k$. Show that for all n and A, there is an isomorphism

$$H_n(X_1; A) \oplus \ldots \oplus H_n(X_k; A) \cong H_n(X; A)$$

induced by the inclusion maps $X_i \to X$.

Remark 7.5.2. It is in fact true that for any (possible infinite) collection $\{X_i\}_{i\in I}$ of spaces, we have

$$H_n(\coprod_{i\in I} X_i; A) \cong \oplus_{i\in I} H_n(X_i; A).$$

This fact cannot be proven based on the axioms I've told you so far: It is a new axiom of homology. However, we do not utilize it in this course, so I will not make much mention of it (though it is very important if you proceed in topology). When we see a model for homology toward the end of this course, you can prove the above isomorphism directly.