Mayer-Vietoris Sequence

2/7/24

Questions:

Proposition: Let X be a topological space. Fix $A \subset X$ and endow A with the subspace topology. Then, the inclusion function
$$i : A \to X$$
$$a \mapsto a$$
is continuous.

Proof: Need to show $\forall$ open $U \subset X$, $i^{-1}(U)$ is open in A.

Well, $i^{-1}(U) = \{ a \in A \mid i(a) \in U \}$
$= \{ a \in A \text{ s.t. } a \in U \}$
$= A \cap U$

By defn. of subspace topology, $i^{-1}(U)$ is thus open (in A).

Last Time:

$\Rightarrow f, g$ have same co/domain $X \to Y$ and are cts.

homotopic

Fact: If $f \sim g$, then $f_* = g_*$ ← induced maps on homology

$\exists\ H : X \times [0,1] \to Y$ cts s.t.
$H(-, 0) = f$
$H(-, 1) = g$

homotopy equivalent

Corollary: If $X \simeq Y$, then $H_n(X; A) \cong H_n(Y; A)$ (for all $n \geq 0$, A)

A <u>homotopy equivalence</u> is a continuous function $f : X \to Y$ s.t. $\exists\ g : Y \to X$ continuous for which $fg \sim id$, $gf \simeq id$

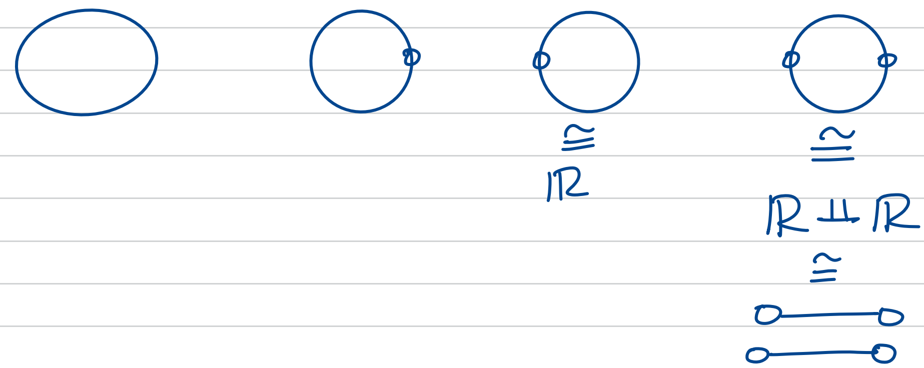
Example: $pt \simeq \mathbb{R}^n$
$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$

Today:

Fact: $\forall n, A$, $H_n(\emptyset; A) \cong 0 = \{0\} \cong \{e\}$ (trivial abelian group)

Warm-Up: Fix two subsets $U, V \subset X$
Have cts. functions:
$$i_U : U \to X, \quad i_V : V \to X$$
$$j_U : U \cap V \to U, \quad j_V : U \cap V \to V$$
$(U \cap V \subseteq U)$

Example: $X = S^1$ $\quad U \quad V \quad U \cap V$



Hence, we have maps (group homomorphisms)

$$((j_u)_*\gamma, (j_v)_*\gamma) \to H_n(U;A) \oplus H_n(V;A) \to H_n(X;A)$$

$$(\alpha, \beta) \longmapsto (i_u)_*\alpha - (i_v)_*\beta$$

$\gamma \in H_n(U \cap V; A)$

Because of minus sign, composition is 0

$$\gamma \mapsto ((j_u)_*\gamma, (j_v)_*\gamma)$$
$$\mapsto (i_u)_*(j_u)_*\gamma - (i_v)_*(j_v)_*\gamma$$
$$= (i_u \circ j_u)_*\gamma - (i_u \circ j_u)_*\gamma = 0$$

Example: $X = \mathbb{R}$
$U = (-1, 3)$
$V = (0, 4)$

$$(-1, 3) \xrightarrow{i_u} \mathbb{R}$$

$A = \mathbb{Z}$ $\qquad\qquad$ $A = \mathbb{Z}$

$$\mathbb{Z} \cong H_n(U;A) \xrightarrow[(i_u)_*]{\cong} H_0(X;A) \cong \mathbb{Z}$$

$$1 \longmapsto \pm 1$$
$$n \longmapsto \pm n$$

So, $H_0(U;\mathbb{Z}) \oplus H_0(V;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \ni (n, m)$

$\downarrow$

$H_0(X;\mathbb{Z}) \cong \mathbb{Z}$

$\pm n + \pm m$ $\quad$ send a pair to their sum

Rest of Today: Fix A, $H_n(X) := H_n(X;A)$

<u>Theorem</u>: (Existence of Mayer-Vietoris Sequence)

Fix $U, V \subset X$ s.t. (i) U, V are open
(ii) $U \cup V = X$

Then, $\forall n \geq 1$, $\exists$ gp. homomorphisms

$$H_n(X) \xrightarrow{\delta} H_{n-1}(U \cap V)$$

such that in the sequence

$$
\begin{aligned}
&H_n(U\cap V)\to H_n(U)\oplus H_n(V)\to H_n(X) \\
\xrightarrow{\delta}\ &H_{n-1}(U\cap V)\to H_{n-1}(U)\oplus H_{n-1}(V)\to H_{n-1}(X) \\
\to\ &\cdots\cdots \qquad\qquad H_1(X) \\
\to\ &H_0(U\cap V)\to H_0(U)\oplus H_0(V)\to H_0(X)\to 0
\end{aligned}
$$

Such that the image of each map is the kernel of the next map

Example:

$X = pt \amalg pt$

$U\cap V = \emptyset$ $\qquad U = \{p\}$ $\qquad V = \{q\}$

both U and V are open subsets

Start M-V sequence at the bottom

$$
\begin{aligned}
&\overset{0}{H_2(U)}\oplus \overset{0}{H_2(V)} \xrightarrow{i_2} \overset{0}{H_2(X)} \\
\xrightarrow{\delta_2}\ &\overset{0}{H_1(U\cap V)}\to \overset{0}{H_1(U)}\oplus \overset{0}{H_1(V)} \to \overset{0}{H_1(X)} \\
\to\ &\underbrace{H_0(\overset{0}{U\cap V})}_{\emptyset}\to \underbrace{H_0(\overset{A}{U})}\oplus H_0(\overset{A}{V}) \xrightarrow{i_0} H_0(X)\xrightarrow{\varepsilon} 0
\end{aligned}
$$

image of (i_2) = $\{0\}$: i_*'s subtracted from before

b/c domain of $(i_2) = 0$

= kernel of δ_2

$\to$ trivial, hence δ_2 is injective

Since codomain $(\delta_2)\cong 0$, we conclude $H_2(X)\cong 0$

so $H_0(\emptyset)=0$ by earlier fact

since U is a single point, its homology is

$$H_n \cong \begin{cases} A, & \text{if } n=0 \\ 0, & \text{if } n\geq 1\end{cases}$$

- Because $\ker(\varepsilon) = \text{image}(i_0)$ and codomain $(\varepsilon) = 0$ we conclude i_0 is a surjection
- i_0 is an injection because $\ker(i_0) = \text{image}(j_0) = \{0\}$
- Hence, $H_0(X)\cong A\oplus A$

Note: Typically, the image of, say, $H_2(U)\oplus H_2(V)$ will not be all of, say, $H_2(X)$, but it will be a subset of it

Prop: $$H_n(pt\amalg pt; A)\cong \begin{cases} A\oplus A, & n\geq 0 \\ 0, & \text{otherwise}\end{cases}$$

Definition: A sequence of group homomorphisms is called *exact* if $\forall$ maps f in the sequence, $\ker(f) = \text{image(preceding map)}$

Example: The MV Sequence is exact

Do: 7.5.1