Reading 6

Homotopies and homotopy equivalence

In this lecture, we'll introduce a notion that was not obviously important at the beginnings of Topology: the notion of *homotopy*.

By now, the idea of homotopy is indispensable for the foundations of topology. (In fact, there is even a subfield of topology called *homotopy theory*, which in the last few decades has become quite influential in broad areas of math.)

Definition 6.0.1. Fix two topological spaces X and Y. Let f and g be continuous functions from X to Y. A *homotopy* from f to g is a continuous function

$$H : X \times [0,1] \to Y$$

such that the function $x \mapsto H(x,0)$ equals f, and the function $f \mapsto H(x,1)$ equals g. Put another way,

$$H(-,0) = f, \qquad \text{and} \qquad H(-,1) = g.$$

6.1 Intuition

Imagine a stretched-out rubber band, floating in space. You can imagine this as a function $f : S^1 \to \mathbb{R}^3$ – a configuration of a circle in space. Now watch the movie of the rubber band shrinking. For every time t, this gives a new function $f_t : S^1 \to \mathbb{R}$ – a new way in which the circle (i.e., the rubber band) is sitting inside space.

This movie from time $t = 0$ to time $t = 1$ is an example of a homotopy from the time $t = 0$ function to the $t = 1$ function.

Of course, "continuous" has a technical definition, so this example is mostly for intuition. (We don't write out any formulas or descriptions to justify why the function $S^1 \times [0,1] \to \mathbb{R}, (x,t) \mapsto f_t(x)$ is continuous.) The rigorous proof of this claim would be to model the evolution of a rubber band as the solution to a differential equation, and to argue that any solution to this particular differential equation must be continuous.

6.2 Exercises

Exercise 6.2.1. We let $X = pt$ be a point; we use $*$ to denote the unique element of X. We set $Y = \mathbb{R}^2$. Consider the function

$$H : X \times [0,1] \to Y, \qquad (*, t) \mapsto (1,2) + (2,2)t.$$

(a) For every $t \in [0,1]$, consider the function $H_t : X \to Y$ given by sending $* \mapsto H(*, t)$. Draw the image of H_0, of $H_{1/4}$, of $H_{1/2}$, of $H_{3/4}$, and of H_1.

(b) Is H continuous? Why or why not?

(c) For every $t \in [0,1]$, is the function H_t continuous? Why or why not?

(d) H is a homotopy between two functions – the ones at $t = 0$ and $t = 1$. Articulate the domain and codomain of these two functions, and state what these two functions are.

Exercise 6.2.2. We let $X = S^1$ be the circle – recall that S^1 is the subset of $\mathbb{R}^2$ consisting of elements (x_1, x_2) a unit distance from the origin.

We set $Y = \mathbb{R}^2$.

Consider the function

$$H : X \times [0,1] \to Y, \qquad ((x_1, x_2), t) \mapsto (tx_1, tx_2).$$

(a) For every $t \in [0,1]$, consider the function $H_t : X \to Y$ given by sending $(x_1, x_2) \mapsto H((x_1, x_2), t)$. Draw the image of H_0, of $H_{1/4}$, of $H_{1/2}$, of $H_{3/4}$, and of H_1.

(b) Is H continuous? Why or why not?

(c) For every $t \in [0,1]$, is the function H_t continuous? Why or why not?

(d) H is a homotopy between two functions – the ones at $t = 0$ and $t = 1$. Articulate the domain and codomain of these two functions, and state what these two functions are.

Exercise 6.2.3. We let $X = Y = \mathbb{R}^2 \setminus \{0\}$.

Consider the function

$$H : X \times [0,1] \to Y, \qquad ((x_1, x_2), t) \mapsto (1-t)(x_1, x_2) + t\frac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}}$$

(a) For every $t \in [0,1]$, consider the function $H_t : X \to Y$ given by sending $(x_1, x_2) \mapsto H((x_1, x_2), t)$. Draw the image of H_0, of $H_{1/4}$, of $H_{1/2}$, of $H_{3/4}$, and of H_1. (For values of t strictly between 0 and 1, this exercise is a bit difficult to do in an enlightening way – it may help to fix $x \in X$ and understand its image under various t.)

(b) Is H continuous? Why or why not?

(c) For every $t \in [0,1]$, is the function H_t continuous? Why or why not?

(d) H is a homotopy between two functions – the ones at $t = 0$ and $t = 1$. Articulate the domain and codomain of these two functions, and state what these two functions are.

6.3 Homotopies and homology

Here is one reason to care about homotopies when homology groups are concerned. We do not prove the theorem.

Theorem 6.3.1. Let $f, f' : X \to Y$ be two continuous functions. If f and f' are homotopic, then for every $n \geq 0$ and every abelian group A, f and f' induce the *same* maps on homology:

$$f_* = f'_*.$$

6.4 Homotopy equivalence induces isomorphisms on homology

Theorem 6.3.1 inspires the following definition.

Definition 6.4.1 (Homotopy equivalence)**.** Let $f : X \to Y$ be a continuous function. We say that f is a *homotopy equivalence* if there exist

(i) a continuous function $g : Y \to X$,

(ii) A homotopy $gf \sim \mathrm{id}_X$, and

(iii) A homotopy $fg \sim \mathrm{id}_Y$.

In other words, f is a homotopy equivalence if it admits an inverse "up to homotopy."

Definition 6.4.2. We say two spaces X and Y are *homotopy equivalent* if there exists a homotopy equivalence from X to Y.

Exercise 6.4.3 (Optional)**.** Let $\{X\}$ be some set of spaces. Show that homotopy equivalence is an equivalence relation on the set.

(The only reason we do not say that homotopy equivalence is an equivalence relation "on the set of all spaces" is that there is no such thing as the set of all spaces, for reasons having to do with Russell's paradox.)

Exercise 6.4.4. Using Theorem 6.3.1, show that if $f : X \to Y$ is a homotopy equivalence, then the induced map on homology is an isomorphism.

6.5 Examples of homotopy equivalence

Exercise 6.5.1. Show that for any $k \geq 0$, any function $pt \to \mathbb{R}^k$ is a homotopy equivalence.

Explain why this proves that pt and $\mathbb{R}^k$ have isomorphic homology groups.

Exercise 6.5.2. Take as an axiom that for any abelian group A,

$$H_n(pt; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Prove that, for every $k \geq 0$,

$$H_n(\mathbb{R}^k; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6.5.3. Show that for any $k \geq 0$, the inclusion $S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ is a homotopy equivalence.

Exercise 6.5.4. Let X be any topological space. Show that the inclusion

$$i_t : X \to X \times [0,1], \qquad x \mapsto (x,t)$$

is a homotopy equivalence for any $t \in [0,1]$.

Show also that the inclusion

$$j_t : X \to X \times \mathbb{R}, \qquad x \mapsto (x,t)$$

is a homotopy equivalence for any $t \in \mathbb{R}$.

Exercise 6.5.5. Suppose $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences. Show that $g \circ f$ is a homotopy equivalence.

Exercise 6.5.6. Suppose $f : X \to Y$ is a homeomorphism. Show that f is a homotopy equivalence.