# Homotopy (Equivalence)

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<u>Last time</u>: Downloaded some properties of homology

$\forall n \geq 0$, $\forall$ abelian gp. $A$,

(1) $$\text{Spaces} \to \text{Abelian Groups}$$
$$X \longmapsto H_n(X;A)$$

"The $n^{th}$ homology group of $X$ with coefficients in $A$"

When $A$ is implicit, we often write $H_n(X)$ (for $H_n(X;A)$)

(2) $\forall f: X \to Y$ continuous, we have an "induced" homomorphism $f_*: H_n(X;A) \to H_n(Y;A)$

(*) When $f = id$, $f_* = id$

(3) $(g \circ f)_* = g_* \circ f_*$

That is, $H_n(-;A)$ is a functor.

- Think of $n$ in $H_n$ as representing something in $n^{th}$ dimension i.e. $H_0$ captures something zero dimensional, $H_1$ captures something 1 dimensional, etc.
- Think about homology groups as a cubist painting of the space

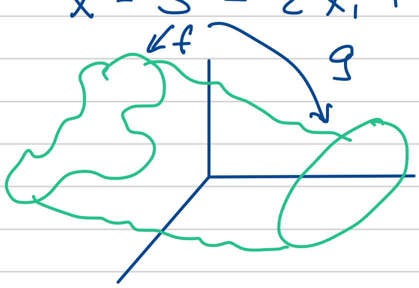
Proposition: If $X \cong Y$ (homeomorphic), then $\forall n \geq 0$, $\forall A$, $H_n(X;A) \cong H_n(Y;A)$

Fact: $$H_n(pt;A) \cong \begin{cases} A, & n = 0 \\ 0, & n \geq 1 \end{cases}$$

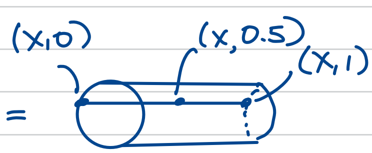
Just take this fact for granted

<u>Definition</u>: Fix two continuous functions $f, g: X \to Y$. A homotopy from $f$ to $g$ is a continuous function
$$H: X \times [0,1] \to Y$$
$$(x,t) \longmapsto H(x,t)$$
such that
$$H(-,0) = f$$
$$H(-,1) = g$$

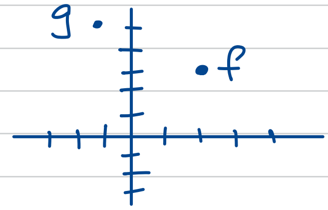
Example: $X = S^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$




$X \times [0,1] =$



Example: $X = \text{pt.}$, $Y = \mathbb{R}^2$
$f: X \to Y$
$* \to (2,3)$
$g: X \to Y$
$* \to (-1,5)$




Define $H: X \times [0,1] \to Y$
$t \longmapsto (1-t)\cdot(2,3) + t\cdot(-1,5)$
At $t=0$, image of function is $(2,3)$
At $t=1$, " " $(-1,5)$
What happens in between? It's a straight line
Points move from f to g as time goes from 0 to 1

Given $f, g: X \to \mathbb{R}^n$
$H(x,t) := (1-t)f(x) + tg(x)$
Any two functions are homotopic if their codomain is $\mathbb{R}^n$

- We say f is <u>homotopic</u> to g if $\exists$ a homotopy from f to g
- "Homotopic to" is an equivalence relation (on the set of continuous functions $X \to Y$)
  (can try to show this as an exercise)

Do: 6.2.1
6.2.2
6.2.3

↳ Continuity - proving continuous using defn's can be tedious here

Cheat: Anything with a formula is probably cts.
Sums, products, composition of functions is cts.

$$\mathbb{R} \times \mathbb{R} \xleftarrow{f} \mathbb{R}$$

$$(t^3, \cos t \sin t) \longleftarrow t$$

Projections $\swarrow x_1$ $\searrow x_2$

$\mathbb{R}$ $\quad$ $\mathbb{R}$
$t^3$ $\quad$ $\cos t \sin t$

Check projections
Universal property of product spaces

What about when you don't have products?

$$\begin{tikzcd} X \times [0,1] \arrow[r, "\cong", "X=pt"'] & {[0,1]} \arrow[r] \arrow[d, "f"'] & \mathbb{R} \arrow[dl, "\tilde{f}"] \\ & Y & \end{tikzcd}$$

Unless you have invented a new topological space, you won't usually prove continuity from the definition → use universal properties

Example: $f: \mathbb{R}^2 \setminus \{0\} \xrightarrow{id} \mathbb{R}^2 \setminus \{0\}$
$(x_1, x_2) \longmapsto (x_1, x_2)$

$g: \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$
$(x_1, x_2) \longmapsto \dfrac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}}$ (vector scaled by its length = unit vector)

Image is $S^1$

Homotopy will take plane and shrink to circle

Define $H: (\mathbb{R}^2 \setminus \{0\}) \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$

$$((x_1, x_2), t) \mapsto (1-t)(x_1, x_2) + t \frac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}}$$

This is the straight line homotopy we saw earlier
Important that this one never crosses origin because dividing by length of vector

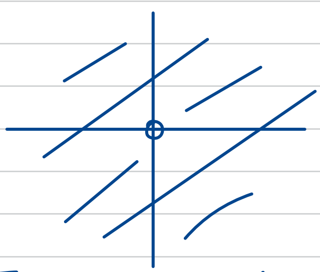


Image at all times $t \neq 1$

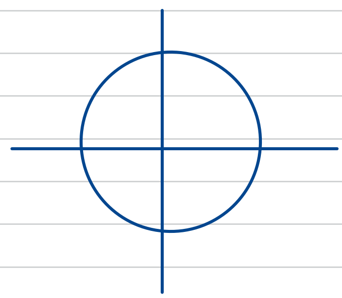


Image at $t = 1$.

homotopic (equivalence relation)

Fact: If $f \sim g$, then $f_* = g_*$

Abelian groups are not sensitive to "wiggles"
You can ask if one function can wiggle into another one

**Definition:** A continuous function $f: X \to Y$ is called a homotopy equivalence if

- $\exists\ g: Y \to X$
- $\exists\ H: X \times [0,1] \to X,\ G: Y \times [0,1] \to Y$

such that

$$\underset{\uparrow \text{composition}}{gf} \underset{H}{\simeq} id_X \quad \text{and} \quad \underset{\uparrow \text{composition}}{fg} \underset{G}{\simeq} id_Y$$

If these are equalities and don't need H, G, then we have a homeomorphism. When we need H and G, then they are "equivalent up to homotopy"

**Proposition:** If $X \underset{\text{homotopy equivalent}}{\simeq} Y$, then $\forall n \geq 0$, $\forall A$,
$$H_n(X; A) \underset{\text{isomorphic}}{\cong} H_n(Y; A)$$

Example: $pt \simeq \mathbb{R}^n \ (\forall n \geq 0)$

Why? $f: pt \to \mathbb{R}^n$, $* \mapsto 0$ $\qquad g: \mathbb{R}^n \to pt$, $x \mapsto *$

$gf = id_{pt}$ $\qquad fg \sim id_{\mathbb{R}^n}$
$\mathbb{R}^n \to \mathbb{R}^n$
$x \mapsto 0$

So, by proposition, we know homology groups of $\mathbb{R}^n$ are homology groups of a point

Example (helpful for HW 3):

$S^1 \simeq \mathbb{R}^2 \setminus \{0\}$

$f: S^1 \to \mathbb{R}^2 \setminus \{0\}$, $x \mapsto x$ $\qquad g: \mathbb{R}^2 \setminus \{0\} \to S^1$, $x \longmapsto \frac{x}{\|x\|}$

$gf = id_{S^1}$ $\qquad fg \sim id_{\mathbb{R}^2 \setminus \{0\}}$