Reading 5

Homology (properties without definitions)

Mathematics is always exciting when different fields of math help each other. The reason we have spent the first few classes reviewing both topology and algebra is that – whether we like it or not – algebra and topology help each other.

In this course, we'll mostly see the way in which algebra helps topology.[1] The main tool we will learn is called *homology*.

5.1 What homology does

Here, in a nutshell, is what homology will be.

Begin by fixing an abelian group A. Then

(I) For every topological space X, homology assigns an infinite family of abelian groups:

$$H_0(X;A), \qquad H_1(X;A), \qquad H_2(X;A), \qquad H_3(X;A), \qquad \ldots.$$

[1] Here are some famous ways in which topology helps algebra: The proof of the fundamental theorem of algebra (that every complex polynomial has at least one root), the classification of semisimple Lie groups and Lie algebras (classifying symmetries that are continuous), concocting new presentations of certain groups (by giving, for example, different presentations of fundamental groups of spaces), and on and on.

Here, each $H_i(X;A)$ is an abelian group. As you can see, there are countably many of them – there is an abelian group for every non-negative integer. These are called the *homology groups of X with coefficients in A*. $H_n(X;A)$ is called the *nth homology group of X* (with coefficients in A).

Indeed, changing the abelian group A can change the above groups! In this class, we will mostly work with $A = \mathbb{F}_2$, and sometimes we will work with $A = \mathbb{Z}$.

In summary: For every space X, for every integer $n \geq 0$ and every abelian group A, homology defines an abelian group $H_n(X;A)$.

(II) For every continuous function $f : X \to Y$ between topological spaces, homology assigns a group homomorphism between homology groups:

$$H_0(X;A) \to H_0(Y;A), \qquad H_1(X;A) \to H_1(Y;A), \qquad , \ldots.$$

Each of these is called the *induced map* on homology, or the map on homology induced by f.[2]

(a) If $f : X \to X$ is the identity function (sending every element x to itself) then the induced map on homology is also the identity function (for every homology group).

(III) Finally, homology does something that is amazing (though it may not be clear why it is so amazing): If $f : X \to Y$ is a continuous function and if $g : Y \to Z$ is another continuous function, the map induced by $g \circ f$ is the composition of the maps induced by g and by f. In other words, induced maps on homology *respect composition*.

Remark 5.1.1. The above properties come up often enough in mathematics that it has a name. We say that (for every n and every A) $H_n(-;A)$ is a *functor* from topological spaces to abelian groups. You won't need to know this terminology for this course.

Notation 5.1.2. We often use the notation

$$H_*(X;A)$$

[2] In general, if two continuous maps are different, then their induced maps may also be different. So the induced map on homology depends on f.

to denote the entire collection of homology groups of X. Or, sometimes we use $H_*(X;A)$ as though $*$ is a variable – a symbol waiting for an integer n to be plugged in.

We also often use the notation

$$f_* : H_*(X;A) \to H_*(Y;A) \qquad \text{and} \qquad f_* : H_n(X;A) \to H_n(Y;A)$$

to denote the induced maps. Confusingly, the "lower-star" notation f_* is ubiquitous (in the literature) for both the settings encoded in the above line.

5.2 Homology is an invariant

Before we get to examples, I want us to prove the following:

Theorem 5.2.1. If $f : X \to Y$ is a homeomorphism, then the maps on homology induced by f are all isomorphisms.

In particular, if two topological spaces X and Y are homeomorphic, then all of their homology groups are isomorphic.

More precisely: Suppose X and Y are homeomorphic. Then for every n and every A, the group homomorphism induced by f $H_n(X;A) \cong H_n(Y;A)$ is an isomorphism.

Put another way, homology groups are an *invariant* of topological spaces. Indeed, take the contrapositive of Theorem 5.2.1.

Corollary 5.2.2. If two spaces have non-isomorphic homology groups – that is, if there exists some n and some A for which $H_n(X;A)$ is not isomorphic to $H_n(Y;A)$ – then the spaces must not be homeomorphic.

Here is a guided series of exercises to help prove the theorem:

Proposition 5.2.3. Let $f : X \to Y$ be a function and $g : Y \to Z$ be a function.

Suppose $g \circ f$ is a bijection. Show that f is an injection and g is a surjection.

Proposition 5.2.4. Let X be a set, and let $f : X \to X$ be the identity function – meaning $f(x) = x$. Show that f is a bijection.

Notation 5.2.5. Let $f : X \to Y$ be a continuous function. In what follows, we let

$$H_n(f) : H_n(X; A) \to H_n(Y; A)$$

be the induced function guaranteed by (II).

Exercise 5.2.6. Suppose that $f : X \to Y$ is a homeomorphism. Consider the following sequence of statements.

(i) Let $f^{-1} : Y \to X$ be the inverse continuous function.

(ii) We know that for every n and A, $H_n(f \circ f^{-1}) = H_n(f) \circ H_n(f^{-1})$.

(iii) Therefore, $H_n(f) \circ H_n(f^{-1})$ is the identity function of $H_n(Y; A)$.

(iv) We conclude that $H_n(f)$ is a surjection.

(v) Using similar reasoning, we see that $H_n(f^{-1}) \circ H_n(f)$ is the identity function of $H_n(X; A)$.

(vi) Therefore, $H_n(f)$ is an injection.

(vii) We conclude that $H_n(f)$ is a bijection.

(viii) Therefore, $H_n(X; A)$ and $H_n(Y; A)$ are isomorphic.

Here is the exercise:

(a) Justify every statement (i) - (viii) above. You will need to use the previous propositions, and the properties of homology I told you about.

(b) Have we proven Theorem 5.2.1? Discuss.

5.3 Examples of homology

In our next few classes, we will discuss techniques for computing homology. But to whet our appetites, I will now give you some examples of what homology looks like.

(In this class, we will mostly focus on the case where the coefficient group A equasl $\mathbb{F}_2$ or $\mathbb{Z}$.)

5.3.1 The point

Let X be a single point. (There is a unique topology on this set – if you haven't seen this before, try it out as an exercise.) Then for any abelian group A, the homology of X is as follows:

$$H_n(pt; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

So, most homology groups are zero, while the 0th (zeroth) homology group is a copy of A. So by plugging in our two favorite examples of A, we get:

$$H_n(pt; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & n = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad H_n(pt; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

(Here, pt is my notation for the point.)

5.3.2 Points

Now let X be a collection of k many disjoint points, with the discrete topology. Concretely, you can take k points inside of Euclidean space (of any dimension) and give this subset the subspace topology. Then we will see that

$$H_n(X; A) \cong \begin{cases} A^{\oplus k} & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

That is, the homology of k points is zero in all degrees except degree $n = 0$. There, the homology group is given by k copies of the coefficient group A. So by plugging in our two favorite examples of A, we get:

$$H_n(X; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2^{\oplus k} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad H_n(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\oplus k} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Combining the above two examples, we see that a collection of k points and a collection of l points are not homeomorphic unless $k = l$. You could have seen this already by noting that k points and l points could not even be in bijection (let alone admit a homeomorphism between them). Likewise, perhaps the easiest way to see that $\mathbb{F}_2^{\oplus k}$ and $\mathbb{F}_2^{\oplus l}$ are not isomorphic is to see that the former has 2^k elements, while the latter has 2^l elements (and hence could not be isomorphic as groups, as they do not even admit a bijection between them).

Notation 5.3.1 (Disjoint union)**.** We write

$$pt \coprod pt$$

for the disjoint union of two points, and

$$pt \coprod \ldots \coprod pt$$

for the disjoint union of k points, where k is how many copies of pt are implied to appear in the above line.

The notation $\coprod$ means "disjoint union," and this terminology may differ slightly from your previous classes. In previous classes, the phrase "disjoint union" referred to a *property* of a union – you can take the disjoint union of two sets that happen to have no intersection.

However, above, we take "disjoint union" to be a new *operation*, which formally treats all its constituents as disjoint. For example,

$$\{A, B\} \bigcup \{A, C\}$$

would normally only have three elements (called A, B, C) but

$$\{A, B\} \coprod \{A, C\}$$

is a set with four elements. One model for this disjoint union is the set

$$\{A', B', A, C\}.$$

Perhaps you get some idea of what I mean by "formally" treating the sets as disjoint.

5.3.3 Euclidean space

It turns out that the homology of $\mathbb{R}^k$ (for any dimension k) is isomorphic to the homology of a point. We will see the computation in one or two weeks, but for now let me just re-iterate the answer:

$$H_n(\mathbb{R}^k; A) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now, here is a more sophisticated fact that should be emphasized in any course on group theory. You should not care *whether* two groups are isomorphic, but if they are, *how*. In other words, if somebody claims that G and H

are isomorphic groups, you should be very curious what group isomorphism that person can construct.

Here, the group isomorphism $H_n(pt; A) \cong H_n(\mathbb{R}^k; A)$ is induced by geometry. Remember that continuous functions induce maps between homology groups[3]. To take advantage of this, choose any element $x \in \mathbb{R}^k$. Then the function

$$pt \to \mathbb{R}^k$$

sending the unique element of pt to x, induces a map on homology groups

$$H_n(pt; A) \to H_n(\mathbb{R}^k; A)$$

for all $n \geq 0$ and for all abelian groups A. It turns out that this map is an isomorphism (for all n and for all abelian groups A).

This example shows that homology is not a "complete" invariant. In other words, there may be two spaces X and Y that have isomorphic homology groups (in fact, the isomorphism may even be induced by a continuous map!) but where X and Y are not homeomorphic.

Example 5.3.2. pt is not homeomorphic to $\mathbb{R}^k$ for any $k \geq 1$. (There couldn't even be a bijection between the two, as $\mathbb{R}^k$ has infinitely many elements.) Regardless, these two spaces have isomorphic homology groups.

5.3.4 The circle

Now let's see our first example of homology groups that are not all concentrated in degree zero.

Let S^1 be the unit circle, endowed with the usual topology (that is, S^1 is endowed with the subset topology inherited from $\mathbb{R}^2$).

Then it turns out

$$H_n(S^1; A) \cong \begin{cases} A & n = 0 \\ A & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

So, if you were to encode this in a table, after plugging in our favorite examples of A, we find:

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[3]See (II)

$$
\begin{aligned}
H_0(S^1;\mathbb{F}_2) &= \mathbb{F}_2 & H_0(S^1;\mathbb{Z}) &= \mathbb{Z} \\
H_1(S^1;\mathbb{F}_2) &= \mathbb{F}_2 & H_1(S^1;\mathbb{Z}) &= \mathbb{Z} \\
H_2(S^1;\mathbb{F}_2) &= 0 & H_2(S^1;\mathbb{Z}) &= 0 \\
H_3(S^1;\mathbb{F}_2) &= 0 & H_3(S^1;\mathbb{Z}) &= 0 \\
\vdots &= 0 & \vdots &= 0
\end{aligned}
$$

Then, because S^1 and $\mathbb{R}^k$ (for any k) have non-isomorphic homology groups, S^1 is not homeomorphic to $\mathbb{R}^k$ (for any k).

(This could be proven another way: S^1 is compact by the Heine-Borel theorem, but $\mathbb{R}^k$ is not compact for $k \geq 1$.)

We will take the homology of a point for granted. In the coming class sessions, we'll see how to actually compute homology groups for all the examples above using some axioms for homology. Get excited!