set of homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$

☆ Is there a bijection $\underbrace{Hom(\mathbb{Z},\mathbb{Z})} \cong \mathbb{Z}$?

Yes! How?

↳ Prop: For any group B, the fxn $f: \{\mathbb{Z} \to B\}$ such that $f(i+j) = f(i) + f(j)$ $= \underset{\text{Domain}}{Hom(\mathbb{Z}, B)} \to \underset{\text{Codomain}}{B}$.

This is a universal, defining property of the integers!

What is written in these notes does not quite make sense :-). The function from Hom(Z,B) to B takes a homomorphism f to f(1).

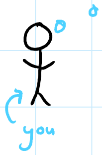
- Let B be $\mathbb{Z}$. Then the statement holds!
- Exercise: Prove the prop.

☆ Homology + its properties

What is homology?

☆ Homology is an assignment:

For any top. space X, any integer $n \geq 0$, and any abelian group A, the nth homology of X w/ coeff. in A denoted $H_n(X, A)$ is an abelian group.



$X \to \{H_n(X,A)\}_{n \geq 0}$

This arrow should be a |—> arrow, not a —> arrow. (Note the vertical butt.)

Each space gets a special sequence of groups!

If $\exists$ some n, A such that $H_n(X,A) \overset{\text{NOT group isomorphic}}{\not\cong} H_n(Y,A)$, then $X \overset{\text{NOT homeomorphic}}{\not\cong} Y$.

Homology groups distinguish top. spaces!

What's purple and commutes? An abelian grape!

This joke needs to be exorcised.

☆ Properties of Homologies:

1) $\forall$ top. space X, $\forall n \geq 0$, $\forall$ abelian group A, we have an abelian group $H_n(X,A)$.

2) $\forall$ continuous fxn $f: X \to Y$, we have group homomorphisms $H_n(X,A) \to H_n(Y,A)$.

NOTE: This map is often denoted $f_*: H_n(X,A) \to H_n(Y,A)$. (this * means an index)

This * does not mean index :-)

We often write $H_*(X,A)$ to mean $\{H_n(X,A)\}$

$H_*(X,A)$ ← this * is an index

f_* ← this * says "push forward": go from X to Y.

2.a) When f is the identity fxn, f_* is also the identity fxn.

$\forall n \geq 0$ and $\forall A$, $id_*: H_n(X,A) \to H_n(X,A)$

$a \mapsto a$

3.) $(\ )_\star$ respects composition.

That is, $\forall$ $f: X \to Y$ and $g: Y \to Z$ that are continuous,

$(g \circ f)_\star = g_\star \circ f_\star$ looks like group homomorphism!

Remark: 1 thru 3 mean that homology is a functor!

Exercise from reading 5:

☆ S.2.3 thru S.2.6

Given $f: X \to Y$ is a homeomorphism.

By defn, f is cont.

f is a bijection.

f^{-1} is cont.

By prop 2, $H_n(f)$ and $H_n(f^{-1})$ exist.

$$
\begin{aligned}
H_n(f) \circ H_n(f^{-1}) &= H_n(f \circ f^{-1}) \leftarrow \text{prop 3} \\
&= H_n(id_Y) \\
&= id_{H_n(Y:A)} \leftarrow \text{prop 2a}
\end{aligned}
$$

Any id fxn is a bijection.

Then by S.2.3, $H_n(f)$ is onto, and $H_n(f^{-1})$ is an injection.

Then, use the same reasoning on $H_n(f^{-1}) \circ H_n(f)$ to show $H_n(f)$ is an injection.

Thus, $H_n(f)$ is a bijection.

Therefore, if f is a homeomorphism, then $H_n(f) = f_\star$ are all group isomorphisms.

Corollary (contrapositive):

If $\exists$ n and A such that $H_n(X,A) \neq H_n(Y,A)$, then X is not homeomorphic to Y.

☆ Examples of Homology:

- Let X be a point (we write $X = pt$). X is a set w/ 1 element.

  Then, $\exists$ only 1 topology on X.

For our class, A will always be $\mathbb{F}_2$ or $\mathbb{Z}$.

Then, $H_0(X, \mathbb{F}_2) \cong \mathbb{F}_2$,

and $H_1(X, \mathbb{F}_2) \cong 0$ ...

Then, $H_0(X, \mathbb{F}_2) \cong \mathbb{F}_2$,
and $H_n(X, \mathbb{F}_2) \cong 0 \quad \forall n > 0$.

the 1-element group $\{e\}$.

Also, $H_n(pt, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{when } n=0 \\ 0 & \text{when } n \geq 1 \end{cases}$

- Another example: Let X be the set w/ k points w/ discrete topology, the topology of k distinct pts in $\mathbb{R}^2$.

Specifically, $X = \underbrace{pt \amalg pt \amalg \dots \amalg pt}_{k \text{ times}}$ ← disjoint union

X has k "copies" of pt.

Then $H_n(X, \mathbb{F}_2) \cong \begin{cases} \underbrace{\mathbb{F}_2 \oplus \mathbb{F}_2 \cdots \oplus \mathbb{F}_2}_{k \text{ times}} & \text{when } n=0 \\ 0 & \text{when } n>0. \end{cases}$

Similarly, $H_n(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\oplus k} & n=0 \\ 0 & \text{when } n>0. \end{cases}$

- One more: Let $X = S^1$ (the unit circle in $\mathbb{R}^2$).

Then $H_n(X, \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{when } n=0 \\ \mathbb{F}_2 & \text{when } n=1 \\ 0 & \text{when } n>1. \end{cases}$