Reading 3

In-class exercises: Open covers

All the definitions you need are in Reading 1.

3.1 An open cover of $\mathbb{R}^n$

Exercise 3.1.1. For each $n \geq 1$, we can define the following subsets of $\mathbb{R}^n$:

$$U_1 = \{x \in \mathbb{R}^n \,|\, x_1 < 1\}, \qquad U_2 = \{x \in \mathbb{R}^n \,|\, -1 < x_1 < 1\}, \qquad U_3 = \{x \in \mathbb{R}^n \,|\, x_1 > -1\}.$$

(a) Draw U_1, U_2, U_3 for $n = 1$. (You are drawing three subsets of $\mathbb{R}$.)

(b) Draw U_1, U_2, U_3 for $n = 2$. (You are drawing three subsets of $\mathbb{R}^2$.)

(c) Draw U_1, U_2, U_3 for $n = 3$. (You are drawing three subsets of $\mathbb{R}^3$.)

Exercise 3.1.2. Fix n.

1. Explain why each U_i is an open subset of $\mathbb{R}^n$ (with the standard topology). You may use facts from previous class sessions, if you like.

2. Explain why the collection $\{U_1, U_2, U_3\}$ is an open cover[1] of $\mathbb{R}^n$ (with the standard topology).

[1] In fact, U_2 is "unnecessary" in the following sense: You can show that $\{U_1, U_3\}$ is an open cover if you like. I insert U_2 because I want you to get used to this particular triplet U_1, U_2, U_3 of open subsets.

3.2 An open cover of S^1

Exercise 3.2.1. Let $X = S^1$ be the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$.

(a) Draw a picture of X.

(b) For every $i \in \{1, 2, 3\}$, let $V_i := U_i \cap X$ (where U_1, U_2, U_3 are the subset of $\mathbb{R}^2$ defined above). Draw V_1 and V_2 and V_3.

Exercise 3.2.2. Let $X = S^1$ as above, and endow X with the subspace topology inherited from $\mathbb{R}^2$.

(a) Explain why each of V_1, V_2, V_3 is an open subset of X.

(b) Explain why the collection $\{V_1, V_2, V_3\}$ is an open cover of S^1.

3.3 An open cover of S^2

Exercise 3.3.1. Let $X = S^2$ be the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

(a) Draw a picture of X.

(b) For every $i \in \{1, 2, 3\}$, let $V_i := U_i \cap X$ (where U_1, U_2, U_3 are the subset of $\mathbb{R}^3$ defined above). Draw V_1 and V_2 and V_3.

Exercise 3.3.2. Let $X = S^2$ as above, and endow X with the subspace topology inherited from $\mathbb{R}^3$.

(a) Explain why each of V_1, V_2, V_3 is an open subset of X.

(b) Explain why the collection $\{V_1, V_2, V_3\}$ is an open cover of S^2.

3.4 An open cover of S^k

Exercise 3.4.1. Let $X = S^k \subset \mathbb{R}^{k+1}$ be the set $\{(x_1, \ldots, x_{k+1}) \mid x_1^2 + x_2^2 + \ldots x_{k-1}^2 + x_k^2 = 1\}$.

(a) For every $i \in \{1, 2, 3\}$, let $V_i := U_i \cap X$ (where U_1, U_2, U_3 are the subset of $\mathbb{R}^{k+1}$ defined above for $n = k + 1$). Explain why each of V_1, V_2, V_3 is an open subset of X.

(b) Explain why the collection $\{V_1, V_2, V_3\}$ is an open cover of S^k.

(c) Why do you think Hiro didn't ask you to draw S^k and the U_i and the V_i?

(d) Do you want to try to find a way to "draw" or visualize them anyway?

3.5 Bonus problems

Exercise 3.5.1. Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. For any subset $A \subset X$, define $V_i := U_i \cap A$ for every $i \in I$. Explain why the collection $\{V_i\}_{i\in I}$ is an open cover of A.

Exercise 3.5.2. Let $f : X \to Y$ be a continuous function, and let $\{W_i\}_{i\in I}$ be an open cover of Y. Explain why the collection $\{f^{-1}(W_i)\}_{i\in I}$ is an open cover of X.

Exercise 3.5.3. (a) Show that $\{V_1, V_3\}$ forms an open cover of S^1.

(b) Consider $S^1 \times S^1$ as a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$. Show that the collection

$$\{V_1 \times V_1, V_1 \times V_2, V_2, \times V_1, V_2 \times V_2\}$$

is an open cover of $S^1 \times S^1$.

(c) Let $T^m = S^1 \times \ldots \times S^1$ be the Cartesian product of m copies of S^1. Consider X as a subspace of $\mathbb{R}^2 \times \ldots \times \mathbb{R}^2 \cong \mathbb{R}^{2m}$. Using the previous part of this exercise as inspiration, exhibit an open cover of T^m consisting of 2^m open subsets.

(d) Show that $\{S^1 \times V_1, S^1 \times V_2\}$ is an open cover of $S^1 \times S^1$.

(e) Using the previous part of this exercise as inspiration, exhibit an open cover of T^m consisting of two open subsets.