Reading 2

In-class exercises: Open subsets of $\mathbb{R}^n$

All the definitions you need are in Reading 1.

2.1 A useful fact

Proposition 2.1.1. Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with the usual (standard) topology. Fix finitely many continuous functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ and a collection of real numbers $a_1, \ldots, a_k$. Then the set

$$\{x \in \mathbb{R}^n \,|\, \text{For every } i = 1, \ldots, k,\ f_i(x) < a_i\}$$

is an open subset of $\mathbb{R}^n$.

Remark 2.1.2. Usually, Proposition 2.1.1 is the easiest way to start identifying, and producing examples of, open subsets of $\mathbb{R}^n$.

We begin getting to know some examples.

Exercise 2.1.3. For simplicity, let's take $n = 2$ in this exercise.

Based on your knowledge from multivariable calculus (Calc III) write out ten examples of continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Here are five to get you started:

(1) $f_1(x_1, x_2) = x_1^2 + x_2^2 - 3$.

(2) $f_2(x_1, x_2) = x_1$.

(3) $f_3(x_1, x_2) = 3x_1 - 5x_2$.

(4) $f_4(x_1, x_2) = 3x^2 - y + 2$.

(5) $f_5(x_1, x_2) = y - \cos(x)$.

Exercise 2.1.4. For concreteness, let's label your ten functions from Exercise 2.1.3 by $f_1, f_2, \ldots, f_9, f_{10}$. Now, choose real numbers $a_1, \ldots, a_{10}$ – it doesn't matter which ones you choose, just make sure they are concrete.

For each whole number i between 1 and 10, draw the set of all $x \in \mathbb{R}^2$ for which $f_i(x) < a_i$. (Depending on the function and real number, this may be very difficult – do the ones you can.)

Exercise 2.1.5. Using Proposition 2.1.1, and using the functions f_i and a_i from previous exercises, draw some open subsets of $\mathbb{R}^2$. There are many combinations you could take – in fact, even with just 10 choices of f_i and a_i, there are over 1,000 combinations – so don't draw more than 20.

2.2 Prove it

Exercise 2.2.1 (Some basic facts.)**.** Prove as many of the following statements as you feel the need to:

1. For any real number a, the open interval $(-\infty, a)$ is an open subset of $\mathbb{R}$ (with the standard topology).

2. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then the preimage of any open subset of $\mathbb{R}$ is an open subset of $\mathbb{R}^n$.

3. The intersection of finitely many open subsets of $\mathbb{R}^n$ is open.

Exercise 2.2.2. Prove Proposition 2.1.1.

2.3 Bonus problems

Exercise 2.3.1. Prove the following: Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with the usual topology. Fix (possibly infinitely many) continuous functions $\{f_i : \mathbb{R}^n \to \mathbb{R}\}_{i \in I}$ and a collection of real numbers $\{a_i \in \mathbb{R}\}_{i \in I}$. Then the set

$$\{x \in \mathbb{R}^n \,|\, \text{For every } i \in I,\ f_i(x) \leq a_i\}$$

is a closed subset of $\mathbb{R}^n$. (Note that the inequalities are no longer strict inequalities.)

(This requires knowledge of closed subsets and their behaviors.)

Exercise 2.3.2. Show by example that Proposition 2.1.1 is false if one chooses infinitely many continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}$ – that is, show that there is some infinite collection of continuous functions $\{f_i : \mathbb{R}^n \to \mathbb{R}\}_{i \in I}$ and an infinite collection of real numbers $\{a_i \in \mathbb{R}\}_{i \in I}$ so that the set of x for which $f_i(x) < a_i$ for all i is not open. (Hint: In fact, for a well-chosen f, one can take $f_i = f$ for all i, and only vary a_i.)

Exercise 2.3.3. Prove that each of the following subsets of $\mathbb{R}^3$ is open:

(a) $U_1 = \{(x_1, x_2, x_3) \,|\, x_1 < \pi\}$

(b) $U_2 = \{(x_1, x_2, x_3) \,|\, x_3 < e^2\}$

(c) $U_3 = \{(x_1, x_2, x_3) \,|\, x_2 + x_1 < 1\}$

(d) $U_4 = \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 + x_3^2 < 4\}$

Draw each of the above sets.

Exercise 2.3.4. Let U_i be the subsets of $\mathbb{R}^3$ from the previous exercise. For every subset $I \subset \{1, 2, 3, 4\}$ draw the intersection

$$\bigcap_{i \in I} U_i.$$