Reading 1

Rapid overview of point-set topology

Recall that a *topology* on a set X tells us which subsets of X deserve to be called open. A set equipped with a topology is, as a matter of terminology, called a topological space – or a space (for short).

The field of Topology[1] is about studying topological spaces. Since the 20th century (and thanks especially to Emmy Noether's contributions to mathematics as a whole), we now view *continuous functions* as the main tool we have for studying topological spaces. Anachronistically, but correctly, we can view the definition of a topological space as serving to let us define the notion of continuous function:

Fix a function f between two sets X and Y. If both X and Y are given topologies, then f is called *continuous* if preimages of open subsets are open.

The main way to distinguish two spaces is to determine whether they are homeomorphic – we say that a function $f : X \to Y$ is a *homeomorphism* if f is a bijection, f is continuous, and if f^{-1} is continuous. Two spaces are called homeomorphic if there exists a homeomorphism between them, and two spaces are equivalent for all intents and purposes if they are homeomorphic.

[1]In this review, I will write Topology (with a capital T) to mean our field of study. I will write topology (in lower case) to mean the mathematical term – as in a collection of open sets (see Definitions 1.3.1 and 1.3.4).

1.1 What an undergraduate topology class should teach

Now, a typical point-set topology class should teach us (roughly speaking) four things:

1. Examples.
2. Constructions.
3. Facts.
4. Techniques.

1.1.1 Examples

The main example of a topological space you should be familiar with is Euclidean space: For every $n \geq 0$, $\mathbb{R}^n$ is a topological space with the so-called *standard topology*: A subset of $\mathbb{R}^n$ is open if and only if it can be written as a union of open balls. (See Section 1.4 for slightly more details.) You should make sure you understand the dependence on n – a shape that looks open as a subset of $\mathbb{R}^n$ is rarely an open subset of $\mathbb{R}^m$ for $m \neq n$.

The main example of continuous functions you should be familiar with are those between Euclidean spaces. Indeed, from multivariable calculus (Calculus III) you should be able to name lots of continuous functions with domain $\mathbb{R}^m$ and codomain $\mathbb{R}^n$ for $1 \leq m, n \leq 3$. (Of course, in ordinary Calculus, you learn about continuous functions with $m = n = 1$.) As a general rule, if a function is famous enough to have a name – $\cos, \sin, \tan \log, \ln$, exponentiation, polynomials, inverse trig functions, et cetera – then it is continuous.

1.1.2 Constructions

By a construction, I mean a way to produce new examples from old.

The three main constructions for topological spaces I want you to be familiar with are *quotient* spaces, *product* spaces, and *subspaces*. The last one of these is especially important for the beginning of this course – it renders any subset of $\mathbb{R}^n$ into a topological space. (See Definition 1.3.9.) Quotient spaces will be more important in the second half of this course.

Each of these constructions have advantages and disadvantages. For example, it is easy to construct continuous functions out of quotient spaces[2], but sometimes tedious to make continuous functions into them[3]. Dually, it is easy to construct continuous functions into product spaces, but often hard to construction continuous functions out of them. Thankfully, it is easy to construct continuous functions into and out of subspaces.

The main reason all these constructions are taught: they just show up. Indeed, some spaces are most easily described as one construction over another. On the other hand, the spaces that can be described through more than one construction are incredibly rich.

The main construction of continuous functions come in the following flavors: *addition and multiplication* (when the functions land in $\mathbb{R}$), *composition*, and *universal properties*. The first two tell you how to make new continuous functions out of known ones – add, multiply, or compose known continuous functions to get new ones. The last construction (universal properties) is the (philosophically) most sophisticated. Universal properties tend to take continuous functions between known objects and create continuous functions between the new objects. I stated earlier that certain constructions of spaces are convenient in different settings. The reason that quotient spaces are convenient for constructing functions out of them is because of the universal property of quotient spaces. It is easy to construct continuous maps into subspaces because of the universal property of product spaces.

1.1.3 Facts

For reasons that only become evident through experience, certain properties of spaces are useful – they identify the salient strategies used in certain proofs. These properties help us to classify spaces into different kinds of "species." Whether a space is *compact* or *Hausdorff*, for example, greatly influences how a space behaves. Here we give three facts that typically appear in a standard course in point-set topology:

1. If a space X is a subset of $\mathbb{R}^n$, it is compact[4] if and only if X is bounded and closed[5] as a subset of $\mathbb{R}^n$. (This is the Heine-Borel Theorem.)

[2]That is, it is easy to construct functions whose domains are quotient spaces.
[3]It is more tedious to construct functions whose codomains are quotient spaces.
[4]When endowed with the subspace topology
[5]See Definition **??**

2. Any subspace of a Hausdorff space is Hausdorff.

3. If $f : X \to Y$ is a continuous bijection from a compact space to a Hausdorff space, f is in fact a homeomorphism.

There are of course many other useful facts in topology; we don't recall more here.

1.1.4 Techniques

These are the hardest to teach. By definition, topology deals with intersections, unions, and preimages. So the rawest techniques in topology involve computations with these ideas. But proper practice would teach you how to work with open covers, bases for topologies, and so forth.

Now, whether these techniques are needed depends on the direction one heads in. In this course, we will build upward – meaning we will use facts and objects of point-set topology, but we will rarely have to prove results using techniques of point-set topology.

Indeed, the kinds of techniques we will use in this course will look very, very different.

1.2 Euclidean space

In our class, we will talk about Euclidean space a lot. Recall that n-dimension Euclidean space is often denoted $\mathbb{R}^n$. By definition, $\mathbb{R}^n$ is the set of all ordered n-tuples

$$(x_1, x_2, \ldots, x_{n-1}, x_n)$$

where each x_i is a real number. If we say that x or y or p is an element of $\mathbb{R}^n$, the ith coordinates of these elements will be denoted as x_i, or as y_i, or as p_i, respectively.

Example 1.2.1. $(\sqrt{2}, \pi, 1, -2, \frac{3}{4})$ is an element of $\mathbb{R}^5$.

Remark 1.2.2. In general, it is very difficult to visualize anything in $\mathbb{R}^4, \mathbb{R}^5, \mathbb{R}^6$, et cetera. Indeed, subsets of $\mathbb{R}^3$ are hard enough to visualize! It is always a good idea to try to visualize something, so you should – but don't be discouraged if you cannot. The power of mathematics is that we can prove things about objects we cannot even see.

As indicated above, we often write an element of $\mathbb{R}^n$ as $(x_1, \dots, x_n)$, with the ith coordinate being denoted by x_i. Note we are not using (x, y) notation for elements of $\mathbb{R}^2$ (though we sometimes might, out of habit). This is because as we go to higher dimensions, we will run out of letters.

Example 1.2.3. Here is an example application of this notation. We can define a function

$$f : \mathbb{R}^4 \to \mathbb{R}, \qquad (x_1, x_2, x_3, x_4) \mapsto 3x_2 - x_1 x_3 + 7x_4^2 - 9.$$

Then, even though we cannot visualize f, we can evaluate it. We have:

- $f(0, 0, 0, 0) = -9$.
- $f(1, 0, 1, 0) = -10$.
- $f(2, 4, 2, 3) = 3 \cdot 4 - 2 \cdot 2 + 7 \cdot 9 - 9 = 12 - 4 + 63 - 9 = 62$.
- $f(\pi, e, \sqrt{2}, 1) = 3e - \pi\sqrt{2} - 2$.

1.3 Basic definitions

For the record, let's list some basic definitions. Along with what you've already read, these definitions will form the foundation for the first two weeks of this course.

Definition 1.3.1. Fix a set X. A *topology* on X is a subset $\mathcal{T}$ of the power set of X satisfying the following conditions:

1. The empty set and X itself are elements of $\mathcal{T}$.
2. For any finite collection of elements of $\mathcal{T}$, their intersection is an element of $\mathcal{T}$.
3. For any (finite or otherwise) collection of elements of $\mathcal{T}$, their union is an element of $\mathcal{T}$.

Definition 1.3.2. A *topological space*, or *space* for short, is a set equipped with a topology on that set.

Notation 1.3.3. As a matter of rigor, a topological space is a pair $(X, \mathcal{T})$ where X is a topology and $\mathcal{T}$ is a topology on X. Regardless, out of sloth, we will often write "let X be a topological space" with the data of $\mathcal{T}$ left implicit in the writing.

Definition 1.3.4. Let X be a topological space. We say that a subset of X is *open* if it is an element of the topology on X.

Definition 1.3.5. Fix a function $f : X \to Y$. Recall that for any subset V of Y, the *preimage* of V is the set of all elements $x \in X$ for which $f(x) \in V$.

Notation 1.3.6. Confusingly, we often write $f^{-1}(V)$ for the preimage of V. This is confusing precisely because "f^{-1}" is not necessarily a function in this notation – after all, f need not be a bijection to define preimage.

Definition 1.3.7. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for every open open subset of Y, its preimage under f is open in X.

Definition 1.3.8 (Open cover)**.** Let X be a topological space. An *open cover* of X is a collection[6] of open subsets $\{U_i\}_{i \in I}$ for which the union $\bigcup_{i \in I} U_i$ is all of X.

Definition 1.3.9 (Subspace topology)**.** Let X be a topological space and fix $A \subset X$. We say that a subset of A is *open* (in the subspace topology) if and only if it it is the intersection of A with some open subset of X.

Definition 1.3.10 (Closed subset)**.** Let X be a topological space. We say that a subset $V \subset X$ is *closed* if the complement of V is open.

Warning 1.3.11. You heard this many times in your point-set topology class, but it is worth repeating: "Open" and "closed" are not opposite notions. A subset can be both closed and open. A subset can be neither closed nor open. A subset can be open and not closed. A subset can be closed and not open.

[6]Note that this collection is indexed by a set I, and I may be finite or infinite.

1.4 Some elaborations on Euclidean space

Let us elaborate a little on the standard topology of $\mathbb{R}^n$ to make sure we are on the same page. Given an element $x \in \mathbb{R}^n$ and a positive real number r, the *open ball centered at x of radius r* is the set

$$\mathrm{Ball}(x, r) := \{y \in \mathbb{R}^n \mid \mathrm{dist}(x, y) < r\}.$$

Put another way, it is the set of all elements of $\mathbb{R}^n$ that are less than distance r away from x. Recall the distance formula:

$$\mathrm{dist}(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}.$$

By taking squares of both sides of the distance formula, the open ball can equivalently be defined as

$$\mathrm{Ball}(x, r) = \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n} (y_i - x_i)^2 < r^2\}.$$

Example 1.4.1. In $\mathbb{R}$, the open ball of radius r centered at x is the open interval $(x - r, x + r)$.

By definition of the standard topology (Section 1.1.1) a subset of $\mathbb{R}$ is open (in the standard topology) if and only if the subset can be written as a union of open intervals of finite width. Here are some examples of open subsets of $\mathbb{R}$:

- The empty set (which is a union of “no” – or zero many – open intervals).
- Any open interval (a, b) for $a < b$.
- Any union of open intervals $\bigcup_{i \in I} (a_i, b_i)$. For example:

  $$\ldots \coprod (-3, -2) \coprod (-2, -1) \coprod (-1, 0) \coprod (0, 1) \coprod (1, 2) \coprod \ldots.$$

  Here, $\coprod$ means “union of sets having no intersections.” That is, this symbol is the symbol for “disjoint union.” Notice this union is allowed to have infinitely many summands.
- Any interval of the form $(-\infty, a)$ or (a, ∞).
- $\mathbb{R}$ itself.