Lecture 25

Taylor Polynomials

25.1 Derivatives of polynomials are easy to compute at certain places

Exercise 25.1.1. Consider the function

$$T(x) = 5 + (x - \pi) + \frac{9}{2}(x - \pi)^2 + \frac{17}{6}(x - \pi)^3 + \frac{19}{24}(x - \pi)^4.$$

This is a degree four polynomial.

(a) Compute $T(\pi)$. **Hint:** Do *not* expand out the powers of $x - \pi$.

(b) Compute $T'(\pi)$. **Hint:** Do *not* expand out the powers of $x - \pi$.

(c) Compute $T''(\pi)$. **Hint:** ... Guess the hint.

(d) Compute the *third* derivative of T at $x = \pi$. (The third derivative is the derivative of the second derivative.) This is sometimes written $T'''(3)$, or sometimes $T^{(3)}(\pi)$.

(e) Compute $T^{(4)}(\pi)$. That is, compute the fourth derivative of T at $x = \pi$.

Possible solution. (a) Plug in $x = \pi$ into the polynomial to obtain:

$$\begin{aligned} T(\pi) &= 5 + (\pi - \pi) + \frac{9}{2}(\pi - \pi)^2 + \frac{17}{6}(\pi - \pi)^3 + \frac{19}{24}(\pi - \pi)^4 \\ &= 5. \end{aligned}$$

(b) As usual, to evaluate the derivative at $x = \pi$ we first take the derivative of T. Without multiplying out the powers of $x - \pi$, we may simply apply the chain rule:

$$T'(x) = 0 + 1 + \frac{9}{2} \cdot 2(x - \pi) + \frac{17}{6} \cdot 3(x - \pi)^2 + \frac{19}{24} \cdot 4(x - \pi)^3.$$

Now if we plug in $x = \pi$, we obtain

$$T'(\pi) = 1.$$

(c) Let us now take the second derivative of T. Without multiplying out the powers of $x - \pi$, we apply the chain rule:

$$T''(x) = 0 + \frac{9}{2} \cdot 2 + \frac{17}{6} \cdot 3 \cdot 2(x - \pi) + \frac{19}{24} \cdot 4 \cdot 3(x - \pi)^2.$$

Now if we plug in $x = \pi$, we obtain

$$\begin{aligned} T'(\pi) &= 0 + \frac{9}{2} \cdot 2 + \frac{17}{6} \cdot 3 \cdot 2(\pi - \pi) + \frac{19}{24} \cdot 4 \cdot 3(\pi - \pi)^2. \\ &= 9. \end{aligned}$$

(d) Let us now take the third derivative of T. Without multiplying out the powers of $x - \pi$, we apply the chain rule:

$$T''(x) = 0 + \frac{17}{6} \cdot 3 \cdot 2 + \frac{19}{24} \cdot 4 \cdot 3 \cdot 2(x - \pi).$$

Now if we plug in $x = \pi$, we obtain

$$\begin{aligned} T'(\pi) &= \frac{17}{6} \cdot 3 \cdot 2 \\ &= 17. \end{aligned}$$

(e) Let us now take the fourth derivative of T. Without multiplying out the powers of $x - \pi$, we apply the chain rule:

$$T''(x) = \frac{19}{24} \cdot 4 \cdot 3 \cdot 2.$$

Now if we plug in $x = \pi$, we obtain

$$\begin{aligned} T'(\pi) &= \frac{19}{24} \cdot 4 \cdot 3 \cdot 2 \\ &= 19. \end{aligned}$$

□

Exercise 25.1.2. Suppose you have a degree four polynomial of the form

$$T(x) = b_0 + b_1(x-a) + \frac{b_2}{2}(x-a)^2 + \frac{b_3}{3!}(x-a)^3 + \frac{b_4}{4!}(x-a)^4$$

where $a, b_0, b_1, b_2, b_3, b_4$ are real numbers.

(Some notes:

- Remember that 4! is a "factorial." It is a shorthand for the expression $4 \times 3 \times 2 \times 1$. Likewise, $3! = 3 \times 2 \times 1$.

- If it helps, you can choose to replace b_0, b_1, and so forth with concrete numbers like 12 or π. But I want you to get practice reasoning without making those kinds of substitutions. The important point here is that $a, b_0, b_1, b_2, b_3, b_4$ are *not* numbers that change with x; they are constants.

End of notes.)

(a) Compute $T(a)$.

(b) Compute $T'(a)$.

(c) Compute $T''(a)$.

(d) Compute $T^{(3)}(a)$.

(e) Compute $T^{(4)}(a)$.

Exercise 25.1.3. Suppose somebody tells you they have a function $f(x)$, and that

- $f(0) = 1$.

- $f'(0) = 0$.

- $f''(0) = -1$.

- $f^{(3)}(0) = 0$.

- $f^{(4)}(0) = 1$.

(a) Can you find a degree four polynomial $T(x)$ such that

- $f(0) = T(0)$,

- $f'(0) = T'(0)$,

- $f''(0) = T''(0)$,
- $f^{(3)}(0) = T^{(3)}(0)$, and
- $f^{(4)}(0) = T^{(4)}(0)$?

(b) Would you expect the graphs of $T(x)$ and $f(x)$ to be related in any way? Why or why not?

25.2 The definition of a Taylor polynomial

Definition 25.2.1. Let f be a function, and choose a real number a. The *nth degree Taylor polynomial* of f at a is the degree n polynomial T_n satisfying

- $T(a) = f(a)$,
- $T'(a) = f'(a)$,
- $\ldots$,
- $T^{(n)}(a) = f^{(n)}(a)$.

In other words, T_n is the polynomial whose value, derivative, second derivative, $\ldots$, and nth derivative at a all agree with those of f at a.

Based on the previous page, we know that the nth degree Taylor polynomial can be written as

$$T_n(x) = b_0 + b_1(x-a) + \frac{b_2}{2}(x-a)^2 + \ldots + \frac{b_n}{n!}(x-a)^n$$

where $b_1 = f'(a)$, $b_2 = f''(a)$, $\ldots$, $b_n = f^{(n)}(a)$. For example, the coefficient in front of $(x-a)^4$ is given by $\frac{f^{(4)}(a)}{4!}$.

And, on the previous page, you were finding the 4th degree Taylor polynomial to some mystery function f.

25.3 Example: cosine

Let $f(x) = \cos(x)$. We'll find the Taylor polynomials of f at $a = 0$. We'll also plot the graph of T_n next to the graph of $\cos(x)$ to compare.

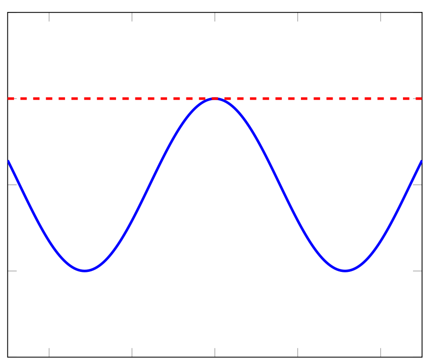
Exercise 25.3.1. Find the degree 0 and degree 1 Taylor polynomials of $f = \cos$ at $a = 0$.

Possible solution. Because $f(0) = 1$ and $f'(0) = 0$, we can find the degree 0 and degree 1 Taylor polynomials as follows:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 + \cos'(0)(x - 0) \\ &= 1 + 0(x - 0) \\ &= 1 \end{aligned}$$

(Note that even though T_1 is called "degree one," it doesn't have a linear term, because the coefficient in front of $(x - a)$ turns out to be zero.) □

Here is the graph of T_0 and T_1 (in dashed red) along with the graph of $\cos(x)$ (in solid blue):



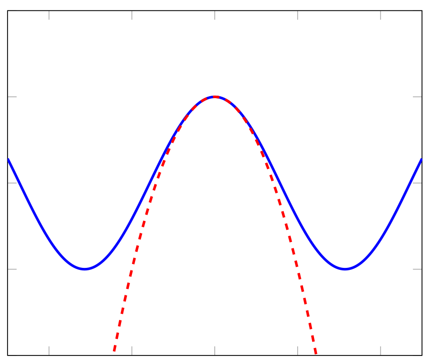
Exercise 25.3.2. Find the degree 2 and degree 3 Taylor polynomials of $f = \cos$ at $a = 0$.

Possible solution. Because $f''(0) = -1$ and $f^{(3)}(0) = 0$, we can find the degree 2 and degree three Taylor polynomials as follows:

$$\begin{aligned}
T_2(x) &= 1 + 0(x-0) + \frac{\cos''(0)}{2}(x-0)^2 \\
&= 1 + \frac{-1}{2}x^2 \\
T_3(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos'''(0)}{6}(x-0)^3 \\
&= 1 + \frac{-1}{2}x^2 + \frac{0}{6}(x-0)^3 \\
&= 1 + \frac{-1}{2}x^2
\end{aligned}$$

(Note that even though T_3 is called "degree three," it doesn't have a degree three term, because the coefficient in front of $(x-a)^3$ turns out to be zero.) □

Here is the graph of $T_2(x)$ (which happens to be the same as $T_3(x)$ in this example) along with the graph of cosine:



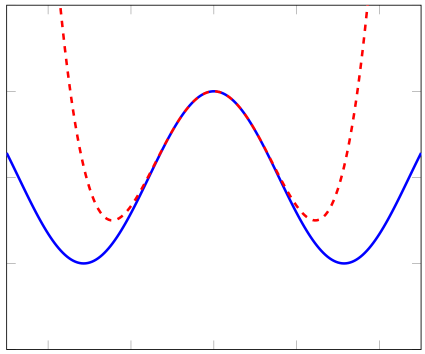
Exercise 25.3.3. Find the degree 4 and degree 5 Taylor polynomials of $f = \cos$ at $a = 0$.

Possible solution.

$$\begin{aligned}
T_4(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos^{(4)}(0)}{24}(x-0)^4 \\
&= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 \\
T_5(x) &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + 0x^5 \\
&= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4
\end{aligned}$$

□

Here is the graph of T_4 next to the graph of cosine:



Question. Are the graphs starting to look similar?

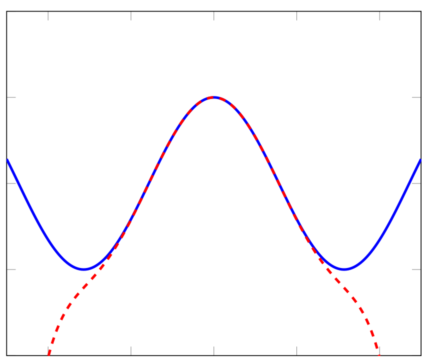
Exercise 25.3.4. Find the degree 6 Taylor polynomial of $f = \cos$ at $a = 0$.

Possible solution.

$$\begin{aligned} T_6(x) &= 1 + 0x + \frac{-1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + 0x^5 - \frac{1}{720}x^6 \\ &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + \frac{-1}{720}x^6 \end{aligned}$$

□

Here is the graph of T_6 next to the graph of cosine:



The take-away: Taylor polynomials allow you to *approximate* a complicated function f by a simpler function (a polynomial), just by knowing the higher derivatives of f at a point a. As you can see from the graphs above, these approximations do a very good job *near* a. Further away from a, the polynomials may behave very differently from f.

25.4 Application: Approximating $\cos(0.5)$

Most of us do not know what $\cos(0.5)$ is off the top of our heads. But we saw in our previous drawings that the Taylor polynomials have graphs that are very similar to the graph of $\cos(x)$ when we are near $a = 0$.

So what if we evaluate $T_n(0)$?

Exercise 25.4.1. (a) For each of the Taylor polynomials $T_0, T_1, T_2, \ldots$ you computed for cos at $a = 0$, compute $T_n(0.5)$. You may use a calculator.

(b) Try comparing these to what your calculator says $\cos(0.5)$ is. I think you'll be pleased!

Possible solution.

- $T_0(0.5) = 1$
- $T_2(0.5) = 1 + \frac{-1}{2}(0.5)^2 = 0.875$
- $T_4(0.5) = 0.87760416667$
- $T_6(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 = 0.87758246528$
- $T_8(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 + \frac{1}{40320}(0.5)^8 = 0.87758256216$

□

25.5 What are Taylor polynomials good for, and when?

Taylor polynomials do a great job of taking a complicated function f, and producing a *polynomial* that very closely approximates f. The input of the number a guarantees a polynomial that closely approximates f *near* a. And in general, the bigger-degree Taylor polynomial we consider, the better the approximation.

The one word of caution is that you should only consider those a for which you can already compute the (higher) derivatives of f at a. For example, to even write down the 0th degree Taylor polynomial of cos at $a = 1$, you would need to know $\cos(1)$. But this is not a number we know off the top of our heads (and to even compute it would require a lot of effort). Likewise, to write down the first degree Taylor polynomial, we would further need to know the derivative of cos at 1 – that is, you'd need to know $-\sin(1)$. This is again a number we don't know very accurately without some effort.

But now, I want you to believe that things like cos(1) are not magical numbers that calculators produce via an unknown mechanism. You could now go home, and try to approximate cos(1) by taking higher- and higher-degree Taylor polynomials for cos at $a = 0$. How cool is that?

Also for fun: Check out this website to have fun with Taylor polynomials for different functions:

`https://www.geogebra.org/m/s9SkCsvC`.