Lecture 24

L'Hôpital's rule and practice with limits

24.1 Warm-up exercises

Exercise 24.1.1. Remember that

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1.$$

For no good reason, let's take the derivative of the top and bottom functions, and then take the limit:

$$\lim_{x \to 0^+} \frac{(\sin(x))'}{(x)'}.$$

What answer do you get?

Exercise 24.1.2. (a) Compute the limit

$$\lim_{x \to \infty} \frac{2x+3}{5x-7}.$$

(Hint: Multiply top and bottom by $1/x$.)

(b) Let's try taking the derivative of the top and bottom function first, and then take the limit. That is, compute

$$\lim_{x \to \infty} \frac{(2x+3)'}{(5x-7)'}.$$

(c) How do your answers compare?

Possible solution. By multiplying top and bottom of the fraction by $1/x$ (which does not change the function outside of $x = 0$) we have:

$$\lim_{x\to\infty} \frac{2x+3}{5x-7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5-\frac{7}{x}} = \frac{\lim_{x\to\infty} 2+\frac{3}{x}}{\lim_{x\to\infty} 5-\frac{7}{x}} = \frac{2+0}{5-0}$$

so we get $2/5$.

For the second part of the problem, we see that

$$(2x+3)' = 2, \qquad (5x-7)' = 5$$

so we have

$$\lim_{x\to\infty} \frac{(2x+3)'}{(5x-7)'} = \lim_{x\to\infty} \frac{(2}{5} = \frac{2}{5}.$$

Our two answers are equal! □

Exercise 24.1.3. Compute the limits

$$\lim_{x\to\infty} \frac{x^2}{1/x} \qquad \text{and} \qquad \lim_{x\to\infty} \frac{(x^2)'}{(1/x)'}.$$

How do your answers compare?

Exercise 24.1.4. What kinds of relationships did you see above between

$$\lim \frac{f(x)}{g(x)} \qquad \text{and} \qquad \lim \frac{f'(x)}{g'(x)}?$$

When were they equal? When were they not?

24.2 L'Hôpital's Rule

The first two exercises were promising, but the last one showed that this trick doesn't always work. Here is a theorem that you may use freely; we won't prove it in this class:

Theorem 24.2.1 (L'Hôpital's Rule)**.** Let f and g be functions that are differentiable. Suppose further that $g'(x)$ does not equal 0 at points close to a.[1]

If

[1] Note $g'(a)$ is allowed to equal zero.

1. $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^+} g(x) = \infty$, or if
2. $\lim_{x\to a^+} f(x) = 0$ and $\lim_{x\to a^+} g(x) = 0$,

then
$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$
provided the righthand side exists.

This holds for lefthand limits as $x \to a^-$ as well, and for limits where a may equal $\pm\infty$.

Roughly, L'Hôpital's Rule says that: If the limits of f and g are both some sort of infinity, or both zero, then the limit of the fraction may be computed by first taking the derivatives of f and g (so long as the limit of the derivatives exist).

Remark 24.2.2. Some textbooks use instead the condition $\lim |f(x)| = \infty$ (and likewise for g), which can seem a little confusing. It turns out this condition is identical to "$\lim f(x) = \infty$ or $\lim f(x) = -\infty$" and likewise for g. (In general, it is very rare for a function to satisfy $\lim |f(x)| = \infty$ without satisfying $\lim f(x) = \infty$ or $\lim f(x) = -\infty$. In fact, such a scenario is impossible if f is continuous and defined for all large values of x. And secretly, we are assuming that f is defined for all large values of x when we compute limits of f as x approaches infinity.)

In either case, these cases all follow from what we've stated above. For example, by the scaling law, $\lim f(x) = -\lim(-f(x))$, so we can always convert a limit equaling $-\infty$ to one equaling ∞.

Warning 24.2.3. The hypothesess of L'Hôpital's Rule are important! (The limits of the denominator and numerator must both agree.) You saw in Exercise 24.1.3 an example where the numerator and denominator had different limits; as a result, the limit of the fraction after taking the derviatives was *different* from the limit of the fraction.

It may also be that the limit of f/g exists, but the limit of f'/g' doesn't exist. Example: $f(x) = x + \cos x$ and $g(x) = x$ and the limit as $x \to \infty$.

Remark 24.2.4. The limits in the statement of L'Hôpital's Rule have no subscripts. This is because I am being lazy. To be explicit: If all the limits are taken at the same point, then the theorem holds.

For example, if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^+} g(x)$ both equal zero, you can apply L'Hôpital's Rule:
$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$
This works for one-sided limits from the left, and for limits at $\pm\infty$.

Remark 24.2.5. As you may have guessed, "L'Hôpital" is a French name. It is pronounced (roughly) "Lo-pee-tahl." You may not be used to the ô; that is, to the little "hat" on top of the *o*. This symbol is called a *circumflex,* and in French, it is often used when a word *used* to have an *s* right after the circumflex. So for example, in the past, the word "L'Hôpital" would have been spelled "L'Hospital." Yes, that's right; this person's name literally translates to "The Hospital."

24.3 Example exercises

Exercise 24.3.1. Compute the following limits.

(a) $\lim_{x\to 0} \frac{e^x-1}{x^2+x}$.

(b) $\lim_{x\to 0} \frac{e^x-x-1}{x^3+x^2}$.

(c) $\lim_{x\to 0} \frac{e^{2x}-2x-1}{x^3+x^2}$.

(d) $\lim_{x\to 0} \frac{e^x}{x^3+x^2}$.

Possible solutions. (a) If we naively try to compute the limits by using the quotient law for limits, we find

$$\frac{\lim_{x\to 0} e^x - 1}{\lim_{x\to 0} x^2 + x} = \frac{e^0 - 1}{0^2 + 0} = \frac{0}{0}.$$

So we obtain something undefined, but we are in luck: Because both the top and bottom limits equal zero, we can try to apply L'Hôpital's Rule. This means we can take the derivative of top and bottom

$$\frac{(e^x - 1)'}{(x^2 + x)'} = \frac{e^x}{2x + 1}$$

then take the limit of this fraction of functions at the same place:

$$\lim_{x\to 0} \frac{e^x}{2x+1} = \frac{\lim_{x\to 0} e^x}{\lim_{x\to 0} 2x + 1} = \frac{e^0}{2 \cdot 0 + 1} = \frac{1}{1} = 1.$$

So, by L'Hôpital's Rule, the limit we seek is equal to 1:

$$\lim_{x\to 0} \frac{e^x - 1}{x^2 + x} = 1$$

(b) We can try to compute the limit by taking the limits of top and bottom of the fraction individually.

$$\frac{\lim_{x\to 0} e^x - x - 1}{\lim_{x\to 0} x^3 + x^2} = \frac{e^0 - 0 - 1}{0^3 + 0^2} = \frac{1-1}{0}. = \frac{0}{0}.$$

Again this method did not yield an answer, but it does tell us we can use L'Hôpital's rule because both the denominator and numerator have equal limits of 0. So we take the derivatives of the top and the bottom:

$$\frac{(e^x - x - 1)'}{(x^3 + x^2)'} = \frac{e^x - 1}{3x^2 + 2x}.$$

If we try to compute the limit of this fraction as x approaches zero by taking the limits of top and bottom, we will again obtain a fraction of $0/0$. So we can *apply L'Hôpital again.*

$$\frac{(e^x - 1)'}{(3x^2 + 2x)'} = \frac{e^x}{6x + 2}.$$

Now we can compute that

$$\lim_{x\to 0} \frac{e^x}{6x+2} = \frac{\lim_{x\to 0} e^x}{\lim_{x\to 0} 6x + 2} = \frac{e^0}{6\cdot 0 + 2} = \frac{1}{2}.$$

After all our work (and having applied L'Hôpital's Rule twice) we find $\lim_{x\to 0} \frac{e^x - x - 1}{x^3 + x^2} = \frac{1}{2}$.

(c) Your answer should be 2, after applying L'Hôpital's Rule twice. You can try graphing this function on a graphing calculator to confirm!

(d) Again we begin by trying to compute the top and bottom limits of the fraction naively:

$$\frac{\lim_{x\to 0} e^x}{\lim_{x\to 0} x^3 + x^2} = \frac{1}{0}.$$

Though we get an undefined answer, it is *not* of the form $0/0$ so we cannot apply L'Hôpital's Rule. Indeed, let's be careful with our work and try evaluating this limit from the right and from the left:

$$\frac{\lim_{x\to 0^+} e^x}{\lim_{x\to 0^+} x^3 + x^2} = \frac{1}{0^+} = \infty.$$

Note that as we plug in a positive value of x to $x^3 + x^2$, the values are positive – so if x approaches 0 from the right, $x^3 + x^2$ approaches zero from the right. And we saw in lab that $\frac{1}{0^+} = \infty$.

On the other hand, the lefthand limit can be computed as follows:

$$\frac{\lim_{x\to 0^-} e^x}{\lim_{x\to 0^+} x^3 + x^2} = \frac{1}{0^-} = -\infty.$$

(The first equality is because if x is a tiny negative number, $x^3 + x^2 < 0$.) We see that the lefthand and righthand limits do not agree, so the limit does not exist at 0.

□

24.4 Practice with limits

Exercise 24.4.1. Evaluate the following limits. Some may involve L'Hôpital's rule; others may not. When you use L'Hôpital's rule, say why you know you can use it (based on the hypotheses of the theorem above).

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$

(e) $\lim_{x\to 0^+} x\ln x$

(f) $\lim_{x\to 0^-} \frac{x}{\sin(x)}$

(g) $\lim_{x\to\infty} xe^x$

(h) $\lim_{x\to-\infty} xe^x$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

24.5 For next time

You should be able to compute all the limits above (and limits similar to them).

(Solutions are on next page if necessary.)

Solutions

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

Evaluating the limit in the numerator and denominator yields

$$\frac{\lim_{x\to(\pi/2)^+}(x-\pi/2)\sin(x)}{\lim_{x\to(\pi/2)^+}\cos(x)} = \frac{(\pi/2-\pi/2)\cdot 1}{0} \tag{24.5.1}$$

This is 0/0, so we can use L'Hôpital's rule.

$$\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)} = \lim_{x\to(\pi/2)^+} \frac{((x-\pi/2)\sin(x))'}{(\cos(x))'} \tag{24.5.2}$$

$$= \lim_{x\to(\pi/2)^+} \frac{\sin(x)+(x-\pi/2)\cos(x)}{-\sin(x)} \tag{24.5.3}$$

$$= \lim_{x\to(\pi/2)^+} \frac{1+0}{-1} \tag{24.5.4}$$

$$= -1. \tag{24.5.5}$$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

Evaluating limits in the numerator and denominator, we obtain ∞/∞, so we can use L'Hôpital's rule.

$$\lim_{x\to\infty} \frac{x}{x^2-1} = \lim_{x\to\infty} \frac{(x)'}{(x^2-1)'} \tag{24.5.6}$$

$$= \lim_{x\to\infty} \frac{1}{2x} \tag{24.5.7}$$

$$= 0. \tag{24.5.8}$$

You also could have solved the original limit without L'Hôpital's Rule: Just divide top and bottom by x.

(c) $\lim_{x\to1^+} \frac{x}{x^2-1}$

We *cannot* use L'Hôpital's Rule here because, when evaluating the limits of the numerator and denominator, we arrive at $1/0$. This is not $0/0$ nor ∞/∞.

But we can still divide top and bottom by x. Then

$$\begin{aligned}
\lim_{x\to1^+}\frac{x}{x^2-1} &= \lim_{x\to1^+}\frac{x}{x^2-1}\cdot\frac{1/x}{1/x} &(24.5.9)\\
&= \lim_{x\to1^+}\frac{1}{x-\frac{1}{x}} &(24.5.10)\\
&= \frac{\lim_{x\to1^+}1}{\lim_{x\to1^+}x-\frac{1}{x}} &(24.5.11)\\
&= \frac{1}{\lim_{x\to1^+}x-\frac{1}{x}}. &(24.5.12)
\end{aligned}$$

When $x > 1$, we know that $x - 1/x$ is positive. So the denominator approaches 0 from the right.

$$= \frac{1}{0^+} = \infty.$$

Here is another way you could have computed this limit. Note that $(x^2 - 1) = (x + 1)(x - 1)$, and we know that $(x - 1)$ is the factor that is causing the denominator to become 0 in the limit. So let's rewrite things in a way we can try to factor out an $(x - 1)$ from the *numerator*, too:

$$\begin{aligned}
\lim_{x\to1^+}\frac{x}{x^2-1} &= \lim_{x\to1^+}\frac{x}{(x-1)(x+1)} &(24.5.13)\\
&= \lim_{x\to1^+}\frac{x-1+1}{(x-1)(x+1)} &(24.5.14)\\
&= \lim_{x\to1^+}\frac{x-1}{(x-1)(x+1)} + \lim_{x\to1^+}\frac{1}{(x-1)(x+1)} &(24.5.15)\\
&= \lim_{x\to1^+}\frac{1}{x+1} + \lim_{x\to1^+}\frac{1}{(x-1)(x+1)} &(24.5.16)\\
&= \lim_{x\to1^+}\frac{1}{1} + \lim_{x\to1^+}\frac{1}{x^2-1} &(24.5.17)\\
&= 1 + \lim_{x\to1^+}\frac{1}{x^2-1}. &(24.5.18)
\end{aligned}$$

Now note that $x^2 - 1$ approaches 0 from the right when $x \to 1^+$, because if $x > 1$, then $x^2 > 1$. So this limit becomes

$$= 1 + \frac{1}{0^+} = 1 + \infty = \infty$$

just as before.

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$ We don't need L'Hôpital's Rule: We see this limit is $1/-\infty = 0$. Note that we couldn't have used L'Hôpital's Rule anyway.

(e) $\lim_{x\to0^+} x\ln x$

Let's rewrite this limit:

$$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+} \frac{\ln x}{1/x} \tag{24.5.19}$$

$$= -\lim_{x\to0^+} \frac{-\ln x}{1/x}. \tag{24.5.20}$$

You don't really need to insert the minus sign, but I did so to see that $\lim_{x\to0^+} -\ln x = \infty$ and $\lim_{x\to0^+} = \infty$; this shows we can apply L'Hôpital's Rule. Applying said rule, we find

$$-\lim_{x\to0^+} \frac{-\ln x}{1/x} = -\lim_{x\to0^+} \frac{(-\ln x)'}{(1/x)'} \tag{24.5.21}$$

$$= -\lim_{x\to0^+} \frac{-1/x}{-1/x^2} \tag{24.5.22}$$

$$= -\lim_{x\to0^+} \frac{1/x}{1/x^2} \tag{24.5.23}$$

$$= -\lim_{x\to0^+} \frac{1/x}{1/x^2} \cdot \frac{x^2}{x^2} \tag{24.5.24}$$

$$= -\lim_{x\to0^+} \frac{x}{1} \tag{24.5.25}$$

$$= -\frac{0}{1} \tag{24.5.26}$$

$$= 0. \tag{24.5.27}$$

(f) $\lim_{x\to0^-} \frac{x}{\sin(x)}$

We already know that $\lim_{x\to0} \frac{\sin x}{x} = 1$, so we can use that to say

$$\lim_{x\to0^-} \frac{x}{\sin(x)} = \lim_{x\to0^-} \frac{1}{\frac{x}{\sin(x)}} \tag{24.5.28}$$

$$= \frac{1}{\lim_{x\to0^-} \frac{x}{\sin(x)}} \tag{24.5.29}$$

$$= \frac{1}{1} \tag{24.5.30}$$

$$= 1. \tag{24.5.31}$$

You could also use L'Hôpital's Rule to arrive at

$$\lim_{x\to 0^+} \frac{1}{\cos(x)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

(g) $\lim_{x\to\infty} xe^x$ You don't need L'Hôpital's Rule here; we plainly see that the limit is given by $\infty \cdot \infty = \infty$.

(h) $\lim_{x\to-\infty} xe^x$ This gets trickier, because we find (taking naive limits)

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} x \lim_{x\to-\infty} e^x = (-\infty)\cdot(0)$$

which is undefined. So let's rewrite this limit as a fraction:

$$\lim_{x\to-\infty} xe^x == \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{24.5.32}$$

$$= \frac{\lim_{x\to-\infty} x}{\lim_{x\to-\infty} e^x} \tag{24.5.33}$$

$$= -\frac{\lim_{x\to-\infty} -x}{\lim_{x\to-\infty} e^x} \tag{24.5.34}$$

$$= -\frac{\infty}{\infty} \tag{24.5.35}$$

which means we can use L'Hôpital's Rule on this limit. We find:

$$\lim_{x\to-\infty} xe^x == \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{24.5.36}$$

$$= \frac{\lim_{x\to-\infty}(x)'}{\lim_{x\to-\infty}(e^x)'} \tag{24.5.37}$$

$$= \frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty} e^x} \tag{24.5.38}$$

$$= \frac{1}{\lim_{x\to-\infty} e^x} \tag{24.5.39}$$

$$= \frac{1}{0^+} \tag{24.5.40}$$

$$= \infty. \tag{24.5.41}$$

$$\tag{24.5.42}$$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

Evaluating the numerator and denominator limits, we obtain ∞/∞, so we can use L'Hôpital's Rule. Then we end up with

$$\lim_{x\to\infty} \frac{\ln 5 \cdot 5^x}{2x}.$$

Taking limits of top and bottom, again we find ∞/∞. So we can use L'Hôpital's Rule *again*. Then we find

$$\lim_{x\to\infty} \frac{(\ln 5)^2 5^x}{2}.$$

The limit of this expression is clearly ∞.

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

This problem is the same work as above, but you use L'Hôpital's Rule three times.