Lecture 23

Limits involving ∞

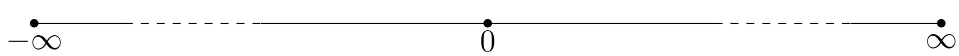
Remark 23.0.1. We will be using the symbol ∞ a lot. This symbol stands for "infinity." I want you to know that the way we use ∞ in calculus class is only *one* way to talk about ∞ in mathematics.

For example, there are "infinitely many" integers; this notion of infinity answers the question "how many?". The "how many" notion is subtly, but definitely, different from the notion of ∞ that we'll use in calculus, which is more dynamical—our notion answers the question "are our numbers eventually getting bigger than any value we specify?"

23.1 Infinity, in our class

23.1.1 Where are ∞ and $-\infty$?

This is controversial among some calculus instructors; but in my class, we will somtimes treat ∞ and $-\infty$ as though they are "numbers." In fact, you can imagine that I've added two ends to the number line:



So for example, between 0 and ∞ lies *every* positive real number. Between $-\infty$ and 0 lies every negative real number. ∞ is larger than any number; $-\infty$ is lesser than any number.

Remark 23.1.1. This should give you some idea for what it means to *approach* infinity. It means that, for any point T on the real line, you eventually surpass and stay larger than T. Likewise, for you to approach $-\infty$ means that, for any number T

on the real line, you eventually become more negative than, and stay more negative than, T.

Indeed, ∞ and $-\infty$ will show up in two ways when we study limits, just as other numbers do. For example, the number 5 shows up in two different ways below:

$$\lim_{x\to 5} f(x) = \pi, \qquad \lim_{x\to\pi} g(x) = 5.$$

Likewise, ∞ can be a *place* where we ask whether a limit exists, and it can also be a *value* that a function approaches.

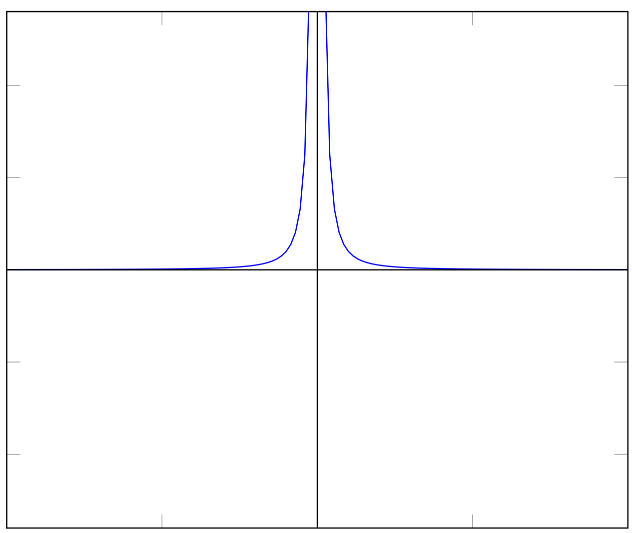
In either case, you can think of approaching ∞ as always approaching ∞ from the left. And many intuitions you have about understanding, say, $\lim_{x\to 5^-}$ – a limit as we approach a location from the left – will hold true for $\lim_{x\to\infty}$.

Likewise, $-\infty$ is always approached from the right. See also Remark 23.6.2.

23.2 Limits equaling infinity

We'll talk about limits equaling ∞ via examples.

Example 23.2.1. Consider the function $f(x) = 1/x^2$. Here's a graph of it:



As you know, $f(x) = 1/x^2$ is *not* defined at $x = 0$. However, does f seem to "want" to do something as x approaches zero?

As you see from the graph, f is "spiking" at $x = 0$, and becoming larger and larger. In fact, if there's a height H that you want to surpass, all you have to do is make sure that x is small enough. For every small-enough x, we know $f(x)$ will be larger than H.

Thus, we say:

$$\lim_{x\to 0} f(x) = \infty.$$

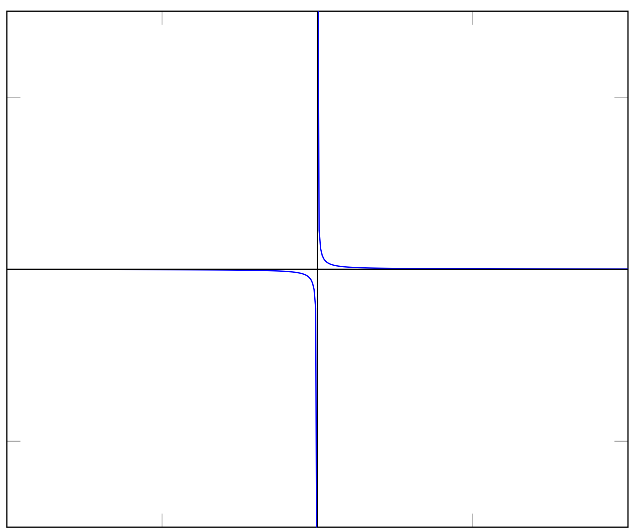
This is our first use of ∞ in calculus class!

Warning 23.2.2. So what does it mean for the limit to equal infinity? It turns out for the value of a function f at a to equal infinity does *not* mean that values keep increasing and increasing in some naive sense.

Rather, it means: For any height H you want to surpass, if you are close enough to a, you are guaranteed that the value of f is above H.

We can talk about left and right limits equaling ∞, too.

Exercise 23.2.3. Consider the function $f(x) = 1/x$. Here's a graph of it:



Evaluate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

Now evaluate $\lim_{x\to 0} f(x)$.

Possible solutions. As you can see, as we approach the origin from the *right*, the graph of f is spiking upward again. We can talk about this righthand limit:

$$\lim_{x\to 0^+} f(x) = \infty.$$

However, as we approach $x = 0$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

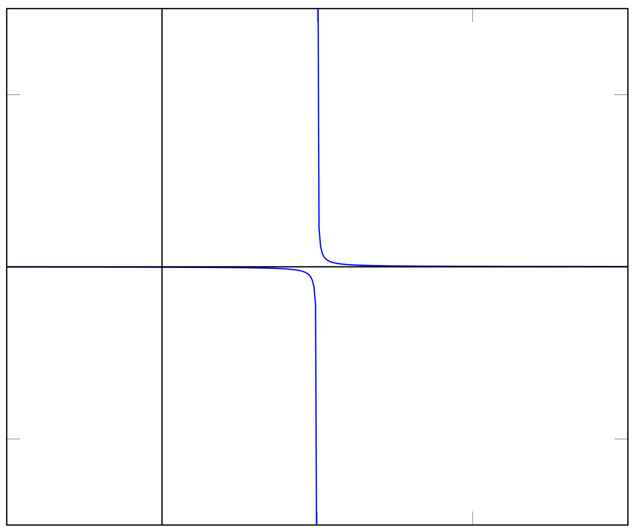
$$\lim_{x \to 0^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x \to 0} f(x) \text{ does not exist.}$$

□

Exercise 23.2.4. Consider the function $f(x) = 1/(x - 0.2)$. Here's a graph of it:



Evaluate

$$\lim_{x \to 0.2^+}, \qquad \lim_{x \to 0.2^-}, \qquad \lim_{x \to 0.2}.$$

Possible solution. As you can see from the graph, as we approach 0.2 from the *right*, the graph of f is spiking upward again. So

$$\lim_{x \to 0.2^+} f(x) = \infty.$$

However, as we approach $x = 0.2$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

$$\lim_{x \to 0.2^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x \to 0.2} f(x) \text{ does not exist.}$$

□

Remark 23.2.5. There is nothing special about 0.2. In fact, for any real number C, we have that

$$\lim_{x \to C^+} \frac{1}{x - C} = \infty, \qquad \lim_{x \to C^-} \frac{1}{x - C} = -\infty, \qquad \lim_{x \to C} \frac{1}{x - C} \text{ does not exist.}$$

23.3 Arithmetic with ∞ and $-\infty$

Of course, you should be able to add/subtract/multiply/divide numbers. Here are the basic rules you need to remember; they are what you would have guessed. (Below, remember that ∞ and $-\infty$ are *NOT* real numbers.)

- Addition and multiplication are still commutative.

Here are the rules involving addition and subtraction:

- If x is a real number, $x + \infty = \infty$ and $x + (-\infty) = (-\infty)$.[1]
- $\infty + \infty = \infty$ and $(-\infty) + (-\infty) = (-\infty)$.

When taking products and quotients, we must never involve zero with $\pm\infty$:

- If x is a *positive* real number, $x \times \infty = \infty$ and $x \times (-\infty) = (-\infty)$.
- If x is a *negative* real number, $x \times \infty = -\infty$ and $x \times (-\infty) = \infty$.
- If x is a *positive* real number, $\infty/x = \infty$ and $(-\infty)/x = (-\infty)$.
- If x is a *negative* real number, $\infty/x = -\infty$ and $(-\infty)/x = \infty$.
- If x is a real number with $x \neq 0$, then $x/\infty = 0$ and $x/(-\infty) = 0$.
- $\infty \times \infty = \infty$ and $\infty \times (-\infty) = (-\infty)$ and $(-\infty) \times (-\infty) = \infty$.

[1] In particular, $\infty - x = \infty$ and $(-\infty) - x = (-\infty)$.

Undefined expressions: Finally, just as you cannot divide a real number by zero, there are certain operations that are undefined when involving $\pm\infty$:

- ∞ - ∞ and $-\infty + \infty$ are undefined.
- ∞/∞ and $-\infty/\infty$ and $\infty/(-\infty)$ and $(-\infty)/(-\infty)$ are all undefined.
- $0 \times \infty$ and $0 \times (-\infty)$ are undefined.
- $0/\infty$ and $0/(-\infty)$ and $\infty/0$ and $(-\infty)/0$ are undefined.

Remark 23.3.1 (A note to educators)**.** That one can do "arithmetic" with $\pm\infty$ is a reflection that the operations $+, -, \times$ (and $\div$) usually defined on $\mathbb{R}\times\mathbb{R}$ (and $\mathbb{R}\times(\mathbb{R}\setminus\{0\})$) extends continuously to a subset of the compactification $[-\infty,\infty]\times[-\infty,\infty]$.

23.4 Limit rules, revisited (this time with ∞)

Once you know how to add/multiply/divide/subtract with ∞, and once you know the basic limits, you can begin to compute limits of more complicated functions.

Here are the basic limit laws for infinity; they are like the old ones, just with more caveats about being careful:

1. (**New:** Limits of $1/(x-C)$). For any real number C, we have that

$$\lim_{x\to C^-}\frac{1}{x-C} = -\infty, \qquad \text{and} \qquad \lim_{x\to C^+}\frac{1}{x-C} = \infty.$$

   (Make sure to take a look at Example 23.2.4 if you haven't yet.)

2. (Scaling law). When the righthand side is defined, for any real number m, we have

$$\lim_{x\to a} mf(x) = m\lim_{x\to a} f(x).$$

   New point of caution: The righthand side is undefined if $m = 0$ and if $\lim_{x\to a} f(x) = \pm\infty$.

3. (Puncture law). If $f(x) = g(x)$ away from a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

4. (Product law) We have that

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

New point of caution: Importantly, the righthand side is not defined when multiplication is not defined—for example, $0\cdot\infty$ is undefined for us. When the righthand side is undefined, you have to *try something different* from the product rule to determine the limit.

5. (Quotient law) We have that

$$\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

New point of caution: Importantly, the righthand side is not defined when division is not defined—for example, $0/\infty$ is undefined. When the righthand side is undefined, you have to *try something different* from the quotient rule to determine the limit.

Remark 23.4.1. Limit laws also work for one-sided limits! This is a good thing. For example,

$$\lim_{x\to a^+}(f(x)\cdot g(x)) = \lim_{x\to a^+} f(x)\cdot \lim_{x\to a^+} g(x).$$

Exercise 23.4.2. Using your knowledge of (one-sided) limits of $1/x$, the product law, and arithmetic with ∞, establish that

$$\lim_{x\to 0}\frac{1}{x^2} = 0.$$

This is an example you should memorize the result of.

Possible solution. It suffices to compute both one-sided limits, and to show that they are the same. Here's one:

$$\lim_{x\to 0^+}\frac{1}{x^2} = \lim_{x\to 0^+}(\frac{1}{x}\cdot\frac{1}{x}) \tag{23.4.1}$$

$$= \lim_{x\to 0^+}\frac{1}{x}\cdot\lim_{x\to 0^+}\frac{1}{x} \tag{23.4.2}$$

$$= \infty\cdot\infty \tag{23.4.3}$$

$$= \infty. \tag{23.4.4}$$

The first line is just algebra. The next line is using the product rule for one-sided limits. Then we are using the fact that we know already the one-sided limits for $1/x$. The last line follows from our rules about arithmetic with ∞.

And here's the other one-sided limit:

$$\begin{aligned}
\lim_{x\to 0^-} \frac{1}{x^2} &= \lim_{x\to 0^-} (\frac{1}{x} \cdot \frac{1}{x}) && (23.4.5)\\
&= \lim_{x\to 0^-} \frac{1}{x} \cdot \lim_{x\to 0^-} \frac{1}{x} && (23.4.6)\\
&= (-\infty) \cdot (-\infty) && (23.4.7)\\
&= \infty. && (23.4.8)
\end{aligned}$$

In sum, we see that both one-sided limits agree, so $1/x^2$ has a limit at 0. We can conclude:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

□

Remark 23.4.3 (Did I do something wrong if I get a non-sense answer?)**.** The first approach most of us take at evaluating limits is trying to "plug the limit in." For example, to compute the previous limit, you may have just tried to set x equal to 0. The issue, of course, is that then you have an expression involving a division by zero.

You are *allowed* to get stuck like that. Just know that when you end up at a division by zero, you've simply tried one thing that does not work. So it is time to try another thing.

This happens a lot with limits. The first few things we try often don't work. So we just have to try another way.

Exercise 23.4.4. Compute

$$\lim_{x\to 0^+} \frac{1+x}{4x^2}.$$

Possible solution. The fastest approach is to use the product law:

$$\lim_{x\to0^+}\frac{1+x}{4x^2} = \lim_{x\to0^+}\frac{1+x}{1}\cdot\lim_{x\to0^+}\frac{1}{4x^2} \tag{23.4.9}$$
$$= 1\cdot\lim_{x\to0^+}\frac{1}{4x^2} \tag{23.4.10}$$
$$= 1\cdot\frac{1}{4}\lim_{x\to0^+}\frac{1}{x^2} \tag{23.4.11}$$
$$= \frac{1}{4}\lim_{x\to0^+}\frac{1}{x^2} \tag{23.4.12}$$
$$= \frac{1}{4}\infty \tag{23.4.13}$$
$$= \infty. \tag{23.4.14}$$

The first line is the product law. (23.4.11) is computing the limit of $\frac{1+x}{1}$. (This is something you already knew how to do.) (23.4.12) is the scaling law. The next line is algebra. (23.4.13) follows from our knowledge of the limit of $1/x^2$ at 0. The last line is arithmetic using ∞.

Here is a different, very tedious approach:

$$\lim_{x\to0^+}\frac{1+x}{4x^2} = \lim_{x\to0^+}\left(\frac{1}{4x^2}+\frac{x}{4x^2}\right) \tag{23.4.15}$$
$$= \lim_{x\to0^+}\frac{1}{4x^2}+\lim_{x\to0^+}\frac{x}{4x^2} \tag{23.4.16}$$
$$= 4\lim_{x\to0^+}\frac{1}{x^2}+\lim_{x\to0^+}\frac{x}{4x^2} \tag{23.4.17}$$
$$= 4\cdot\infty+\lim_{x\to0^+}\frac{x}{4x^2} \tag{23.4.18}$$
$$= \infty+\lim_{x\to0^+}\frac{x}{4x^2} \tag{23.4.19}$$
$$= \infty+\lim_{x\to0^+}\frac{1}{4x} \tag{23.4.20}$$
$$= \infty+\frac{1}{4}\lim_{x\to0^+}\frac{1}{x} \tag{23.4.21}$$
$$= \infty+\frac{1}{4}\cdot\infty \tag{23.4.22}$$
$$= \infty+\infty \tag{23.4.23}$$
$$= \infty. \tag{23.4.24}$$

The first line is algebra, and the next line is the addition rule. Note that we don't know whether the sum will be well-defined[2] at this stage, but we proceed crossing our fingers. Then I kept simplifying the lefthand term in the summation, knowing that $\lim 1/x^2 = \infty$ and using the scaling law. Line (23.4.20) follows from the puncture law. Then I use the scaling law, and then my knowledge of $\lim_{x\to 0^+} \frac{1}{x}$. The last few lines are following the arithmetic of ∞. □

Exercise 23.4.5. Let's compute

$$\lim_{x\to 3^+} \frac{5x}{x-3}.$$

Possible solution. We have the following string of equalities:

$$\begin{aligned}
\lim_{x\to 3^+} \frac{5x}{x-3} &= \lim_{x\to 3^+} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3} && (23.4.25)\\
&= 15 \cdot \lim_{x\to 3^+} \frac{1}{x-3} && (23.4.26)\\
&= 15 \cdot \infty && (23.4.27)\\
&= \infty && (23.4.28)
\end{aligned}$$

The first equality is the product law for limits—note that we did not know[3] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (23.4.26) is evaluating the limit for $5x$, which we knew how to do already. (23.4.27) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞. □

Exercise 23.4.6. Compute

$$\lim_{x\to 3^-} \frac{5x}{x-3}.$$

[2] For example, if at the end we find a sum of the form $\infty - \infty$, we are at a loss—this expression is not defined.

[3] We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3}$ would yield something non-sensical like $0 \cdot \infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

Proof. We have the following string of equalities:

$$\lim_{x\to 3^-} \frac{5x}{x-3} = \lim_{x\to 3^-} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3} \tag{23.4.29}$$
$$= 15 \cdot \lim_{x\to 3^-} \frac{1}{x-3} \tag{23.4.30}$$
$$= 15 \cdot -\infty \tag{23.4.31}$$
$$= -\infty \tag{23.4.32}$$

The first equality is the product law for limits—note that we did not know[4] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (23.4.30) is evaluating the limit for $5x$, which we knew how to do already. (23.4.31) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞. □

Exercise 23.4.7. Compute—using the limit laws above—the one-sided limits

$$\lim_{x\to 3^-} \frac{\ln x}{x-3} \qquad \text{and} \qquad \lim_{x\to 3^+} \frac{\ln x}{x-3}.$$

Does $\lim_{x\to 3} \frac{\ln x}{x-3}$. exist?

23.5 Limits at ∞

Exercise 23.5.1. Compute

$$\lim_{x\to 0^+} \frac{1}{e^x-1} \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{e^x-1}.$$

Try to think this through *without* using the limit laws above—they won't help.

Does $\lim_{x\to 0} \frac{1}{e^x-1}$ exist?

I want to very carefully walk through this last exercise. How would we compute

$$\lim_{x\to 0^+} \frac{1}{e^x-1}?$$

[4]We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3}$ would yield something non-sensical like $0\cdot\infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

Clearly the denominator is causing us trouble. The quotient law doesn't apply because the limit of the denominator is 0.

However, let's think about what's happening to $e^x - 1$ as x approaches 0 from the right. When $x > 0$, we know that $e^x > e^0$. In other words, $e^x > 1$.

Thus, as x approaches 0 from the right, the denominator remains *positive*, but shrinks to zero. (As x approaches 0 from the right, e^x shrinks in size, and e^x becomes closer and closer to 1 while remaining larger than 1. As a result, $e^x - 1$ becomes closer and closer to 0 while remaining positive.)[5]

So $\frac{1}{e^x-1}$, as we shrink x to 0 from the right, is positive, and growing larger and larger (because we are dividing 1 by smaller and smaller numbers). This intuition suggests

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = \infty.$$

Likewise, as x approaches 0 from the left, e^x is less than 1, but is growing in size to 1. Thus $e^x - 1$ is negative, but approaching 0. So we conclude

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = -\infty.$$

We will see how to compute this more rigorously next time, when we also talk about limits *at* $x = \pm\infty$.

23.6 Examples

Exercise 23.6.1. For each of the following functions, determine whether $f(x)$ "approaches" a particular value as x becomes larger and larger. Drawing a rough sketch of the graph may be helpful.

(i) $f(x) = \frac{1}{x}$.

(ii) $f(x) = 2 + \frac{1}{x}$.

(iii) $f(x) = x$.

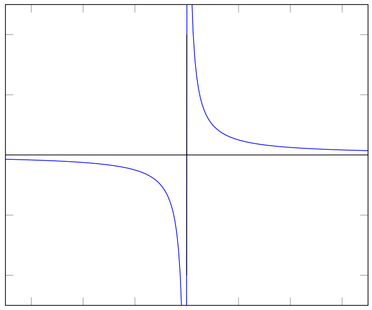
(iv) $f(x) = \sin x$.

(v) $f(x) = \frac{\sin x}{x}$.

(vi) $f(x) = x \sin x$.

[5] Make sure you understand this!

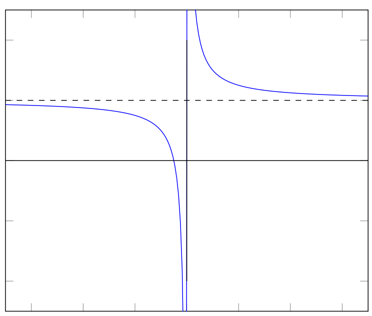
Here are the solutions.



(i) $f(x) = \frac{1}{x}$.

We see in this example that as x becomes bigger and bigger, $1/x$ becomes smaller and smaller; in fact, we can make $1/x$ as close to 0 as we like, so long as x is large enough. We write

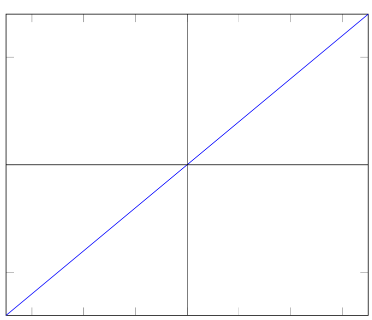
$$\lim_{x\to\infty} \frac{1}{x} = 0.$$



(ii) $f(x) = 2 + \frac{1}{x}$.

We see in this example that as x becomes bigger and bigger, $2 + 1/x$ becomes closer and closer to 2 (the height 2 is drawn in dashes above). In fact, we can make $2 + 1/x$ as close to 2 as we like, so long as x is large enough. We write

$$\lim_{x\to\infty} 2 + \frac{1}{x} = 0.$$

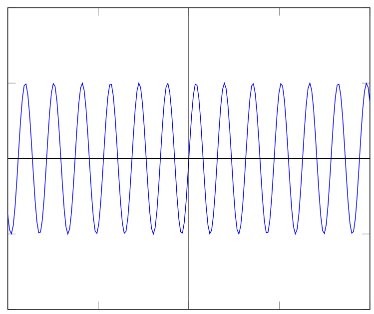


(iii) $f(x) = x$.

In this example, we see that $f(x) = x$ becomes bigger and bigger as x does. In fact, we can say the following: If we want f to be larger than some number T, we just need to ensure that x is larger than T. We say

$$\lim_{x\to\infty} x = \infty.$$

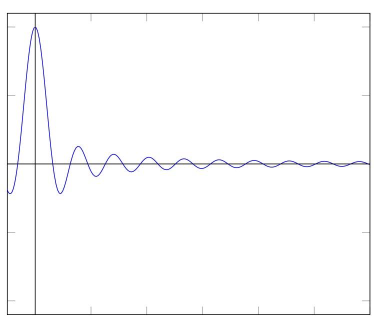
(iv) $f(x) = \sin x$.



This is a tricky example, but we see that no matter how large x is, $f(x)$ could be any number between -1 and 1. And there is *no* big number that guarantees that "so long as x is bigger than this big number, $f(x)$ will be close to some limiting value." Thus, we say

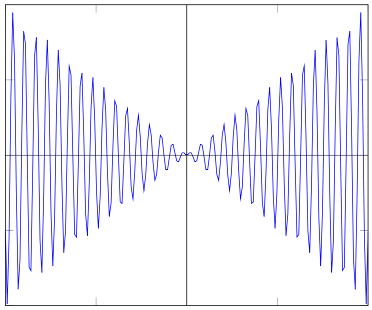
$$\lim_{x\to\infty} \sin(x) \text{ does not exist }.$$

(v) $f(x) = \frac{\sin x}{x}$.



This is different. f still seems to oscillate, but the f is approaching values closer and closer to 0 as x grows. Indeed, we can guarantee f to be ϵ-close to 0 so long as x is large enough. We say

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$



(vi) $f(x) = x \sin x$.

In this example, $f(x)$ displays interesting behavior as x grows larger and larger. f oscillates, and more wildly. Importantly, f does *not* approach infinity as x grows. Here is why: To approach infinity, we must guarantee that for any T, f is larger than T so long as x is large enough. But regardless of how big we require x to be, there is a possibility that $f(x)$ is less than T—in fact, $f(x)$ could even be *negative*!

So we say

$$\lim_{x\to\infty} x\sin(x) \text{ does not exist.}$$

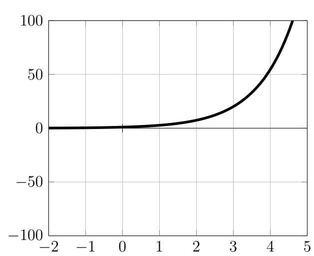
Remark 23.6.2 (Are there no "one-sided" limits at infinity?)**.** You may have noticed we have not dicussed one-sided limits when we approach ∞ or $-\infty$. A better way to think about this is that all limits at ∞ are in some sense one-sided, in that

$$\lim_{x\to\infty} f = \lim_{x\to\infty^-} f.$$

Indeed, there is no sense in which x can approach ∞ "from the right." Likewise, you should think of a limit at $-\infty$ as one-sided, too:

$$\lim_{x\to\infty} f = \lim_{x\to\infty^+} .$$

Example 23.6.3. This is an **important example you need to know**. Look at the graph of e^x:



As x approaches $-\infty$ (i.e., as x goes to the left), the graph approaches the x-axis. So

$$\lim_{x\to-\infty} e^x = 0.$$

As x approaches ∞ (i.e., as x goes to the right), the function grows larger and larger, without bound. So

$$\lim_{x\to\infty} e^x = \infty.$$

23.7 Practice with limit laws for infinities

We will be vague about this, but here it is:

Limit laws work for limits involving ∞^*

with an asterisk: $*$ **so long as all terms are defined**.

Exercise 23.7.1. Compute

$$\lim_{x\to\infty}(x - x^2).$$

Possible solution. We can try using the addition law. If we do, we find

$$\lim_{x\to\infty}(x - x^2) == \lim_{x\to\infty} x - \lim_{x\to\infty} x^2 \tag{23.7.1}$$

$$= \infty - \infty \qquad \text{!!!!!!!} \tag{23.7.2}$$

The big exclamation marks are a warning: The expression "$\infty - \infty$" is **not defined**. This means that the limit law gives us no information (just like the quotient law is inapplicable when the denominator has limit 0). So we tried, and we failed. That's okay.

Let's try something else: The product law. The key observation is to see that

$$(x - x^2) = x(1 - x).$$

Then we have:

$$\lim_{x\to\infty}(x - x^2) = \lim_{x\to\infty} x(1 - x) \tag{23.7.3}$$

$$= \lim_{x\to\infty} x \cdot \lim_{x\to\infty}(1 - x) \tag{23.7.4}$$

$$= \infty \cdot \lim_{x\to\infty}(1 - x) \tag{23.7.5}$$

$$= \infty \cdot (1 - \infty) \tag{23.7.6}$$

$$= \infty \cdot (-\infty) \tag{23.7.7}$$

$$= -\infty. \tag{23.7.8}$$

□

Exercise 23.7.2. Compute

$$\lim_{x\to\infty}(x - x^2 + 10).$$

Possible solution. We can try using the addition law. If we do, we find

$$\lim_{x\to\infty}(x - x^2 + 10) = = \lim_{x\to\infty}(x - x^2) + \lim_{x\to\infty} 10 \tag{23.7.9}$$
$$= (\lim_{x\to\infty}(x - x^2)) + 10. \tag{23.7.10}$$

But we know this limit in the parentheses! We saw above that the limit was $-\infty$, so we obtain

$$\lim_{x\to\infty}(x - x^2 + 10) = -\infty + 10 = -\infty.$$

□

23.8 Limits at ±∞ for polynomials

In fact, repeating the factoring trick and the addition law, you can conclude the following: **You can compute limits at ∞ for polynomials by looking at the highest degree term**, and these limits will always be $\pm\infty$. For example,

$$\lim_{x\to\infty} 3x^5 + x^4 - 3x^2 = \lim_{x\to\infty} 3x^5 \tag{23.8.1}$$
$$= 3(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x) \tag{23.8.2}$$
$$= 3\infty\cdot\infty\cdot\infty\cdot\infty\cdot\infty \tag{23.8.3}$$
$$= 3\infty \tag{23.8.4}$$
$$= \infty. \tag{23.8.5}$$

The first equality is using the bolded principle above (you need only look at the highest degree term of the polynomial when computing limits at $\pm\infty$). The next line is using the product rule a lot.

As another example,

$$\lim_{x\to-\infty} 2x^6 + x^5 - 3x = \lim_{x\to-\infty} 2x^6 \tag{23.8.6}$$
$$= 2(\lim_{x\to-\infty} x)^6 \tag{23.8.7}$$
$$= 2(-\infty)^6 \tag{23.8.8}$$
$$= 2\infty \tag{23.8.9}$$
$$= \infty. \tag{23.8.10}$$

Again, in the first line, I am using that the limit at $\pm\infty$ of a polynomial is equal to the limit of the highest degree term. Note that I got lazy and wrote $(-\infty)^6$ rather than $(-\infty)\cdot(-\infty)\cdot(-\infty)\cdot(-\infty)\cdot(-\infty)$.

Here is one more examples for your edification:

$$\lim_{x\to-\infty} -4x^3+x^2-x = \lim_{x\to-\infty} -4x^3 \tag{23.8.11}$$
$$= -4\cdot \lim_{x\to-\infty} x^3 \tag{23.8.12}$$
$$= -4\cdot(-\infty)^3 \tag{23.8.13}$$
$$= -4\cdot(-\infty) \tag{23.8.14}$$
$$= \infty. \tag{23.8.15}$$

23.9 Limits at $\pm\infty$ for rational functions

Example 23.9.1. Let's compute

$$\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2}.$$

Let's try using the quotient law. We get

$$\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2}. = \frac{\lim_{x\to\infty} x^3+x+1}{\lim_{x\to\infty} 3x^3-3x^2} \tag{23.9.1}$$
$$= \frac{\infty}{\infty}. \tag{23.9.2}$$

This is undefined! So we can't use the quotient law—at least in the way we've used it. We failed, like we've failed before. That's okay. We keep trying.

Here's a wonderful trick: Let's divide top and bottom of the function in question by x^3. Then we obtain:

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{3\frac{x^3}{x^3}-3\frac{x^2}{x^3}}.$$

Let's follow this til the end:

$$\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{3\frac{x^3}{x^3}-3\frac{x^2}{x^3}} \tag{23.9.3}$$

$$= \frac{\lim_{x\to\infty} \frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{\lim_{x\to\infty} 3\frac{x^3}{x^3}-3\frac{x^2}{x^3}} \tag{23.9.4}$$

$$= \frac{\lim_{x\to\infty} 1+\frac{1}{x^2}+\frac{1}{x^3}}{\lim_{x\to\infty} 3-3\frac{1}{x}} \tag{23.9.5}$$

$$= \frac{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 3-\lim_{x\to\infty}3\frac{1}{x}} \tag{23.9.6}$$

$$= \frac{1+0+0}{3-3\cdot 0} \tag{23.9.7}$$

$$= \frac{1}{3}. \tag{23.9.8}$$

The first line was the "divide top and bottom by x^3" trick, the next was the quotient rule, then we did some algebra. We obtain (23.9.6) using the addition rule, and then we obtain (23.9.6) by evaluating the limits we already knew how to evaluate. The final line is just arithmetic.

Here is the general trick: **When computing limits of rational functions *at* $\pm\infty$, divide the top and bottom by the highest power of x you see in the denominator.**

Exercise 23.9.2. Compute $\lim_{x\to\infty} \frac{x^2+x+1}{3x^3-3x^2}$.

Possible solutions. Here is some work:

$$\lim_{x\to\infty} \frac{x^2+x+1}{3x^3-3x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{3-3\frac{1}{x}} \tag{23.9.9}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{\lim_{x\to\infty} 3-3\frac{1}{x}} \tag{23.9.10}$$

$$= \frac{0+0+0}{3-0} \tag{23.9.11}$$

$$= 0. \tag{23.9.12}$$

□

Exercise 23.9.3. Compute $\lim_{x\to\infty} \frac{x^4+x+1}{3x^2-3x}$.

Possible solutions.

$$\lim_{x\to\infty} \frac{x^4+x+1}{3x^2-3x} = \lim_{x\to\infty} \frac{x^2+\frac{1}{x}+\frac{1}{x^2}}{3-3\frac{1}{x}} \tag{23.9.13}$$

$$= \frac{\lim_{x\to\infty} x^2+\frac{1}{x}+\frac{1}{x^2}}{\lim_{x\to\infty} 3-3\frac{1}{x}} \tag{23.9.14}$$

$$= \frac{\lim_{x\to\infty} x^2+0+0}{3-0} \tag{23.9.15}$$

$$= \lim_{x\to\infty} \frac{x^2}{3} \tag{23.9.16}$$

$$= \frac{1}{3}\lim_{x\to\infty} x^2 \tag{23.9.17}$$

$$= \frac{1}{3}\cdot\infty \tag{23.9.18}$$

$$= \infty. \tag{23.9.19}$$

□

The reason this trick works: When you divide the denominator by the highest power of x you see there, you'll always end up with a denominator that looks like

$$\text{some number} + a\frac{1}{x} + b\frac{1}{x^2} + \ldots (\text{some coefficient})\frac{1}{x^k}.$$

But if we take the limit of this expression as $x \to \pm\infty$, we get the same "some number," because all other terms go to zero. In particular, the denominator is an actual number, so we'll never run into a quotient that's undefined.

23.10 Bonus: Definition of limits at infinity

We've seen some examples of limits at infinity. Here is a definition:

Definition 23.10.1. We say that f *has a limit at* ∞ if there exists a number L such that for every real number ϵ, we can guarantee that "if x is big enough, $f(x)$ is within ϵ of L."

More precisely, we say that f has a limit at ∞ if there exists a number L such that for every real number ϵ, we can find a number F so that[6]

$$x > F \implies |f(x) - L| < \epsilon. \tag{23.10.1}$$

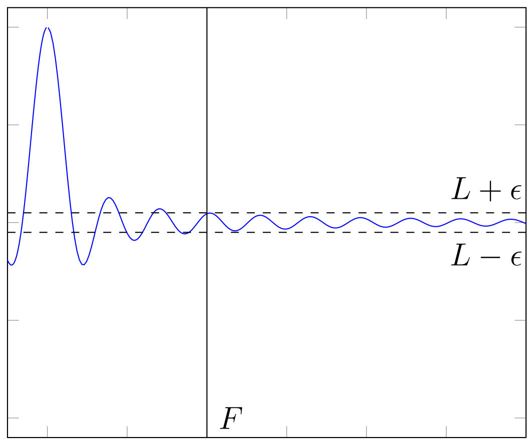
[6] Here, $x > F$ is the mathematical translation of "x is big enough."

(Remember that " $\implies$ " means "implies.")

We call L the *limit of f as x approaches* ∞, and we write

$$\lim_{x\to\infty} f(x) = L.$$

Graphically, (23.10.1) means that so long as our x coordinate is larger than F, our graph of f is within a strip of height 2ϵ centered at L:



You will rarely have to use this definition, but you should know that the definition above provides the mathematical precision necessary to prove things like limit laws for infinity (see next section).

We can also talk about **limits as x approaches** $-\infty$—to find such limits is to ask whether f approaches a particular number as x becomes more and more negative. We write such a limit as

$$\lim_{x\to-\infty} f(x).$$

23.11 For next time

I expect you to be able to compute limits similar to the following:

$$\lim_{x\to 0^+} \frac{1+x}{4x^2}.$$

$$\lim_{x\to 3^+} \frac{5x}{x-3}.$$

$$\lim_{x\to 0}\frac{1}{x^2}$$

$$\lim_{x\to C^-}\frac{1}{x-C} \qquad \text{and} \qquad \lim_{x\to C^+}\frac{1}{x-C} \qquad \text{and} \qquad \lim_{x\to C}\frac{1}{x-C}.$$