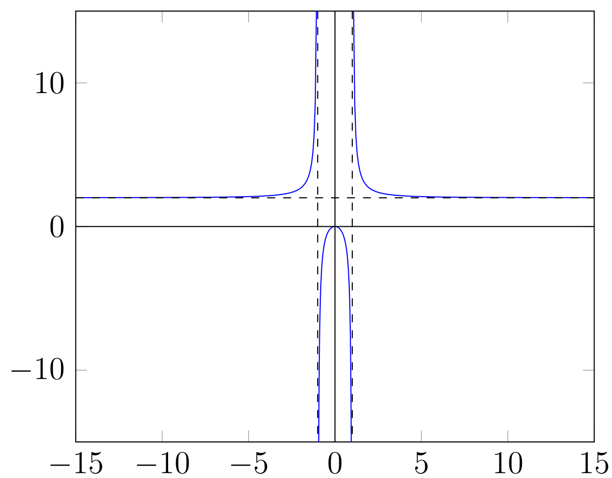
23.12 Lab Topic: Asymptotes

For next lecture, I am going to have you practice finding horizontal and vertical asymptotes. The two things you'll be learning are (i) asymptotes, and (ii) computing limits using $0^{\pm}$ notation—this is a way to improve upon the quotient rule for limits.

First, asymptotes. Here is a vague definition:

Definition 23.12.1. As *asymptote* of a function is a line that approximates the function in some limit.

Example 23.12.2. Below is the graph of the function $f(x) = \frac{2x^2}{x^2-1}$:



I have drawn three dashed lines. Two of them are vertical, at $x = 1$ and $x = -1$. The other is horizontal, at height $y = 2$.

As you can see, as x approaches 1 or -1, the graph of the function begins to look more and more vertical, and the graph becomes near and nearer to the vertical lines . These two vertical lines are called *vertical asymptotes*. They are the lines $x = 1$ and $x = -1$.

You can also see that as x approaches ∞, the graph of f becomes closer and closer to the dashed horizontal line (of height 2). We say that the line $y = 2$ is a *horizontal asymptote* of f.

As it happens, f approaches the same line as x goes to $-\infty$. (This does not need to happen for $y = 2$ to be considered a horizontal asymptote; the graph might approach different horizontal asymptotes at ∞ and at $-\infty$.)

From the way I've described things, you've probably noted the following:

1. We find vertical asymptotes of f by seeing whether $\lim_{x\to a^+} f$ or $\lim_{x\to a^-}$ equals $\pm\infty$ at some a. If this limit does equal $\pm\infty$ at a, then the line $x = a$ is a vertical asymptote of f.

2. We find horizontal asymptotes of f by computing $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, if $\lim_{x\to\infty} f(x) = B$, then f has a horizontal asymptote of height B, because f approaches the horizontal line $y = B$ as x increases. And if $\lim_{x\to-\infty} f(x) = C$, then f also has a horizontal asymptote of height C, because f approaches the horizontal line $y = C$ as x approaches $-\infty$.

23.12.1 The 0^+ and 0^- notation (improving the quotient rule)

There is one more incredibly useful trick for knowing whether a limit is $\pm\infty$. Let me state the fact below as a Lemma; we'll improve upon it shortly:

Lemma 23.12.3. Suppose f is a quotient of two functions, so that

$$f(x) = \frac{g(x)}{h(x)}.$$

Suppose that

(i) $\lim_{x\to a^+} g(x)$ is positive [7], and

(ii) $h(x)$ approaches 0 from the right as x approaches a from the right.

Then

$$\lim_{x\to a^+} f(x) = \infty.$$

Example 23.12.4. Let us study $\lim_{x\to 3^+} \frac{1}{x-3}$. We see that as x approaches 3 from the right (meaning x is always larger than 3, but approaching 3), the expression $x-3$ is always positive, but is approaching 0. In other words, $x-3$ is approaching 0 from the right. Thus the lemma applies, and we conclude

$$\lim_{x\to 3^+} \frac{1}{x-3} = \infty.$$

(We already knew this fact, but we are formalizing it using the Lemma/trick above.)

[7] In particular, it is not equal to zero; but we do allow for this limit to be ∞

Of course, if the denominator approaches 0 from the *left*, then the limit is no longer ∞, but is $-\infty$. So the lemma has different versions for how x approaches a, and how we approach 0 in the denominator. But more importantly, we are *very* lazy, and we don't want to have to say the words "the denominator approaches zero from the right/left" all the time.

So we will have a shorthand notation (Notation 23.12.5)—and it can cause confusion, so be careful. But because of this shorthand notation, the Lemma above will become compressed into some simple equalities, which you'll find in Lemma 23.12.8 below. It's these simple equalities that you'll *actually* be using when writing out work quickly.

Notation 23.12.5 ($0^{\pm}$)**.** Let h be a function, and suppose that as x approaches a from the right, $h(x)$ approaches zero from the right. Then we will write

$$\lim_{x \to a^+} h(x) = 0^+.$$

Likewise, if $h(x)$ approaches zero from the left, we will write

$$\lim_{x \to a^+} h(x) = 0^-.$$

We use the same notation when we have x approach a from the left, too. So we can write things like

$$\lim_{x \to a^-} h(x) = 0^+ \qquad \text{or} \qquad \lim_{x \to a^-} h(x) = 0^-.$$

Warning 23.12.6. Unlike $\pm\infty$, I will **discourage** you from thinking of 0^+ and 0^- as numbers. You should think of 0^+ as shorthand for "approaching zero from the right," and the equality symbol of $\lim_{x \to a^-} h(x) = 0^+$ *not* as an equality of numbers, but a shorthand for saying "0^+ is the way that this limit looks."

Ah, but there are always caveats. See the footnote below.[8]

Example 23.12.7. The following are all correct uses of this notation:

1. $\lim_{x \to 3^+} \frac{1}{x-3} = \frac{1}{0^+}$.
2. $\lim_{x \to 3^-} \frac{1}{x-3} = \frac{1}{0^-}$.

[8] There *is* a system of numbers for which you can think of both $\pm\infty$ and $0^{\pm}$ as legitimate "numbers." But this can be a little confusing at first glance, and a discussion about this can take us very, very far astray, so we won't be exploring this avenue in this class. But I hope you see that a door is cracked open: A door to a place where you can explore new notions of "number" and test your imagination against mathematical truths.

3. $\lim_{x\to 3^+} \frac{1}{3-x} = \frac{1}{0^-}$.

4. $\lim_{x\to 3^-} \frac{1}{3-x} = \frac{1}{0^+}$.

Here is the condensed version of Lemma 23.12.3 above, and of its relatives:

Lemma 23.12.8. Let A be positive. (A can be a number, or it can equal ∞.) Then

$$\frac{A}{0^+} = \infty \qquad \text{and} \qquad \frac{A}{0^-} = -\infty.$$

If instead A is negative, then

$$\frac{A}{0^+} = -\infty, \qquad \text{and} \qquad \frac{A}{0^-} = \infty.$$

(End of Lemma.)

Warning 23.12.9. Expressions like $\frac{A}{0^+}$ *only arise when computing one-sided limits.* I warn you again to only use the above lemma when you are computing certain one-sided limits, and to not think of the equalities above as equalities of numbers, but a shorthand, lazy way of saying (for example) "If the denominator approaches 0 from the right, and if A is positive, then the limit is ∞."

Remark 23.12.10 (A helpful way to think about Lemma 23.12.8)**.** Usually, we can't compute limits when the denominator equals zero. Think of Lemma 23.12.8 as a way of *improving* the quotient rule: So long as we are computing one-sided limits, and so long as we do not get a result that looks like "0/0," we can actually compute limits even when the denominator approaches zero (so long as the denominator only approaches zero from one side).

Exercise 23.12.11. Compute the following limits.

(i) $\lim_{x\to 3^+} \frac{x}{x-3}$.

(ii) $\lim_{x\to 3^-} \frac{x}{x-3}$.

(iii) $\lim_{x\to 3^-} \frac{x^2}{9-x^2}$.

Possible solution. Here are examples of how you can use the above notation to write out the work to compute some limits:

(i)

$$\lim_{x\to 3^+} \frac{x}{x-3} = \frac{\lim_{x\to 3^+} x}{\lim_{x\to 3^+} x-3} \tag{23.12.1}$$
$$= \frac{3}{\lim_{x\to 3^+} x-3} \tag{23.12.2}$$
$$= \frac{3}{0^+} \tag{23.12.3}$$
$$= \infty. \tag{23.12.4}$$

I used the "dividing by 0^+" notation from Lemma 23.12.8 in the last two equalities. Everything else is a straightforward application of limit laws.

(ii)

$$\lim_{x\to 3^-} \frac{x}{x-3} = \frac{\lim_{x\to 3^-} x}{\lim_{x\to 3^-} x-3} \tag{23.12.5}$$
$$= \frac{3}{\lim_{x\to 3^-} x-3} \tag{23.12.6}$$
$$= \frac{3}{0^-} \tag{23.12.7}$$
$$= -\infty. \tag{23.12.8}$$

I used the "dividing by 0^+" notation from Lemma 23.12.8 in the last two equalities. Everything else is a straightforward application of limit laws.

(iii)

$$\lim_{x\to 3^-} \frac{x^2}{9-x^2} = \frac{\lim_{x\to 3^-} x^2}{\lim_{x\to 3^-} 9-x^2} \tag{23.12.9}$$
$$= \frac{9}{\lim_{x\to 3^-} 9-x^2} \tag{23.12.10}$$
$$= \frac{9}{0^+} \tag{23.12.11}$$
$$= \infty. \tag{23.12.12}$$

The important thing to note here is the step from (23.12.10) to (23.12.11). Though x is approaching 3 from the left, $9 - x^2$ is approaching zero *from the right*; this is because as x approaches 3 from the left, x^2 is always less than 9;

so $9 - x^2$ is always positive. In contrast, we have:

$$\lim_{x\to 3^+} \frac{x^2}{9-x^2} = \frac{\lim_{x\to 3^+} x^2}{\lim_{x\to 3^+} 9-x^2} \tag{23.12.13}$$
$$= \frac{9}{\lim_{x\to 3^+} 9-x^2} \tag{23.12.14}$$
$$= \frac{9}{0^-} \tag{23.12.15}$$
$$= -\infty. \tag{23.12.16}$$

$\square$

Now we can put the two new ideas of this packet together.

Exercise 23.12.12. Find all vertical and horizontal asymptotes (if any) of

$$f(x) = \frac{4x^3+3x-2}{3x^2-27}$$

Possible solution. (i) Let's first begin to look for horizontal asymptotes. Remember, this means looking for limits as x approaches $\pm\infty$. And the trick for this for rational functions is to divide the numerator and denominator by the highest power in the denominator. So let's begin our computation doing that, and go on:

$$\lim_{x\to\infty} \frac{4x^3+3x-2}{3x^2-27} = \lim_{x\to\infty} \frac{4x^3+3x-2}{3x^2-27} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \tag{23.12.17}$$
$$= \lim_{x\to\infty} \frac{4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3 - \frac{27}{x^2}} \tag{23.12.18}$$
$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{\lim_{x\to\infty} 3 - \frac{27}{x^2}} \tag{23.12.19}$$
$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3-0} \tag{23.12.20}$$
$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3} \tag{23.12.21}$$
$$= \frac{\lim_{x\to\infty} 4x + \lim_{x\to\infty} 3\frac{1}{x} - \lim_{x\to\infty} 2\frac{1}{x^2}}{3} \tag{23.12.22}$$
$$= \frac{\infty + 0 - 0}{3} \tag{23.12.23}$$
$$= \infty. \tag{23.12.24}$$

Because this limit is not a real number, there is *no* horizontal asymptote that f approaches as x goes to ∞.

An almost identical computation will show that $\lim_{x\to-\infty} f(x) = -\infty$, so that there is no horizontal asymptote that f approaches as x goes to $-\infty$, either. In sum, there are no horizontal asymptotes.

(ii) Finally, let's check for vertical asymptotes. This has to do with checking when the denominator might limit to zero. So we must find when the expression

$$3x^2 - 27$$

could equal zero. This happens when $x^2 = 9$, meaning we must study the limits as x approahces ± 3. All we need to check is, for each of these values, whether *either* of the one-sided limits approaches $\pm\infty$.

So we must compute

$$\lim_{x\to 3^+} \frac{4x^3 + 3x - 2}{3x^2 - 27}. \tag{23.12.25}$$

The numerator becomes

$$4(3)^3 + 3 \cdot 3 - 2 = 4 \cdot 27 - 9 - 2 = 108 - 9 - 2 = 97.$$

On the other hand, the denominator approaches 0 from the right as x approaches 3 from the right. So we have

$$\lim_{x\to 3^+} \frac{4x^3 + 3x - 2}{3x^2 - 27} = \frac{97}{0^+} = \infty.$$

So we have found a vertical asymptote at $x = 3$. (At this point, we are happy with $x = 3$, and we don't need to check the lefthand limit at 3.)

Let's make sure that we have a vertical asymptote at $x = -3$. For example, you will find

$$\lim_{x\to -3^+} \frac{4x^3 + 3x - 2}{3x^2 - 27} = \frac{\text{some non-zero number}}{0^-} = -\infty,$$

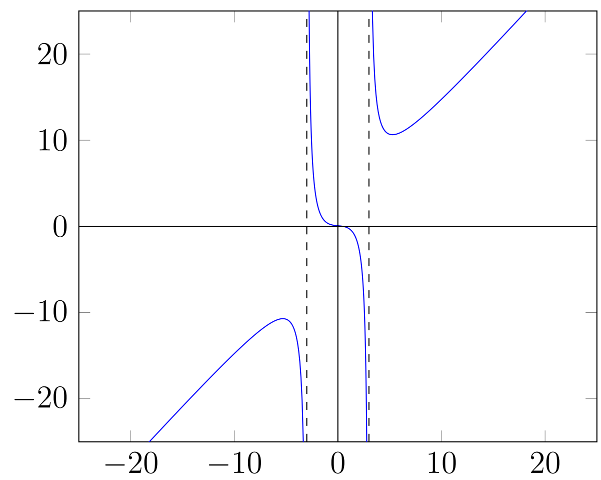
and

$$\lim_{x\to -3^-} \frac{4x^3 + 3x - 2}{3x^2 - 27} = \frac{\text{some non-zero number}}{0^+} = \infty.$$

Computing either of these one-sided limits shows that there is a vertical asymptote at $x = -3$.

To summarize: $f(x)$ **has no horizontal asymptotes, but has two vertical asymptotes at** $x = 3$ **and** $x = -3$**.** □

In case you want to check your answer, here is a graph of the function:



Exercise 23.12.13. Find the vertical and horizontal asymptotes for the following functions:

(a) $f(x) = \frac{1}{x-4}$

(b) $f(x) = \frac{x^2}{x^2-9}$

(c) $f(x) = \frac{x^3}{x^2-9}$