Lecture 22

Application of continuity (Intermediate Value Theorem and Extreme Value Theorem.) Limit example: Derivative of absolute value.

Today, we're going to see that if a function is continuous, some intuitive facts about the function are true. For some motivation behind why this is important, see Section 22.4.

22.1 The Intermediate Value Theorem

22.1.1 Warm-up exercises

Exercise 22.1.1. Consider the function $f(x) = x^2 + 10$. Does this function have a root?

(Recall that a *root* is a value of x for which $f(x)$ equals zero. So, another way to rephrase the question: is there a value of x such that $x^2 + 10$ equals zero?)

Explain.

Exercise 22.1.2. Consider the polynomial function $f(x) = x^5 + 7x^4 - 22x + 19$. (This function is complicated, I know!)

Let me tell you that $f(-10)$ has the value -29,761. Also, $f(3)$ equals 763.

Based on this information, does $f(x)$ have a root?

(This question is *not* asking you to *find* a root; it's asking you whether a root *exists*.)

Explain. Can you explain in such a way where you can ignore/forget how complicated $f(x)$ looks?

A possible solution to Exercise 22.1.1. In the first exercise, there is no root. After all, x^2 is always a non-negative number regardless of x; so adding 10 to x^2 will result in a number larger than zero. Formally, we have that

$$x^2 \geq 0 \text{ (regardless of } x) \qquad \text{and} \qquad 10 > 0 \qquad \text{so} \qquad x^2 + 10 > 0.$$

□

A possible solution to Exercise 22.1.2, based on intuition. There *is* a root; this is because for the graph of this polynomial to begin with a negative height and attain a positive height, it must cross the x-axis at some point. □

Here is a theorem.

Theorem 22.1.3 (Intermediate Value Theorem)**.** Let $f(x)$ be a continuous function, and choose two real numbers a and b with $a < b$[1] and assume f is defined on $[a, b]$. Then for any number N between $f(a)$ and $f(b)$, [2] there is a number c between a and b so that $f(c) = N$.

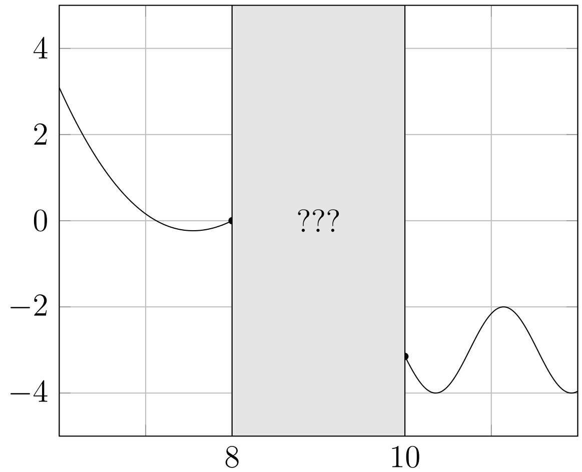
Put another way, on the way from a to b, the graph of f attains every height between $f(a)$ and $f(b)$.

Remark 22.1.4. Sometimes, we abbreviate the Intermediate Value Theorem by "IVT" (especially when we are running out of time on exams or quizzes).

Example 22.1.5. Here is a graph of a function $f(x)$ that your friend began to make, then stopped part-way:

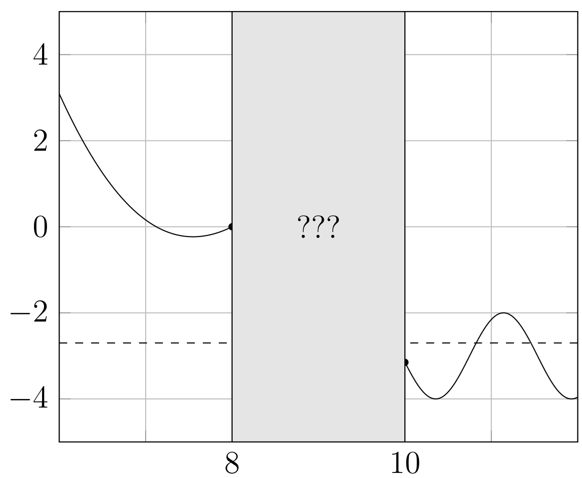
[1] You should imagine these numbers to be on the x-axis.

[2] You should imagine N, $f(a)$, and $f(b)$ to be on the y-axis



So you have no idea what $f(x)$ looks like in the region between 8 and 10. However, you do know that $f(8) = 0$ and $f(10) = -3$. Therefore, *if $f(x)$ is continuous*, then the Intermediate Value Theorem tells you that $f(x)$ must hit (at least) every number between 0 and -3, at least once.[3]

For example, -2.7 is a number between 0 and -3. So, though you *do not know where*, you do know that $f(x)$ must equal -2.7 at *some value* of x between 8 and 10. [4] Here is a pictorial way to think about it:



(22.1.1)

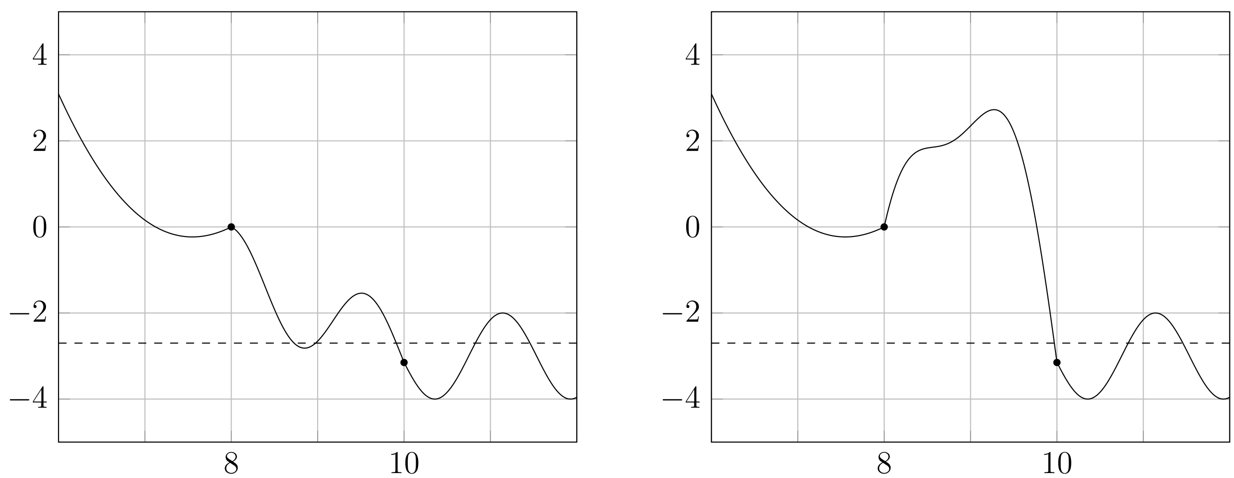
[3]In this example, $a = 8$ and $b = 10$.

[4]In terms of the letters used in Theorem 22.1.3, $N = -2.7$. And c is the *some value* between 8 and 10.

We have drawn, in dashes, the line at height -2.7. Because $f(x)$ is continuous, to get from height 0 to height -3, the graph of $f(x)$ *must* cross over this line at some point in the grey region. We don't know where $f(x)$ crosses the line, but it does so *somewhere* between $x = 8$ and $x = 10$.

Remark 22.1.6. Note that, in Example 22.1.5, the graph of $f(x)$ crosses over the line of height -2.7 *outside* the grey region as well. That's all well and good, but the intermediate value theorem only guarantees something about the *grey region*—i.e., about the region between a and b.

Remark 22.1.7. Here are some examples of continuous functions that could fill in the grey region from (22.1.1):



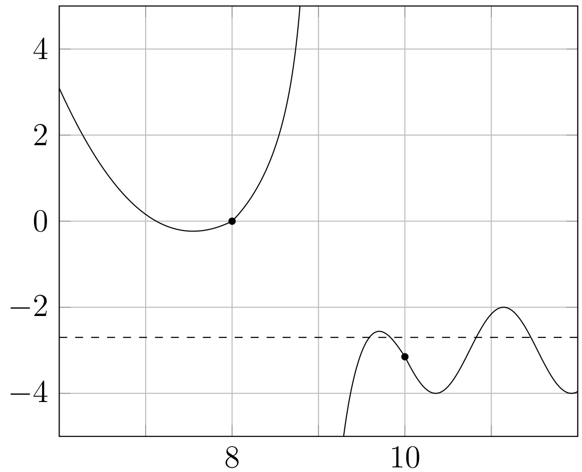
Note that $f(x)$ may attain N at *more than one value of* c. (You can see this graphically in the lefthand example: The graph of $f(x)$ crosses the horizontal line of height $N = -2.7$ three times.)

Note that $f(x)$ *does not need to stay inbetween* $f(a)$ *and* $f(b)$. (You can see this on the righthand example.) That is, even if $a < c < b$, it need *not* be true that $f(c)$ is between $f(a)$ and $f(b)$.

Exercise 22.1.8. Do Exercise 22.1.2 again, using the IVT. Make sure you know what the values of a, b, and N are.

Do you know the value of c?

Exercise 22.1.9. Your friend has taken the mystery function from (22.1.1) and filled it in as follows.

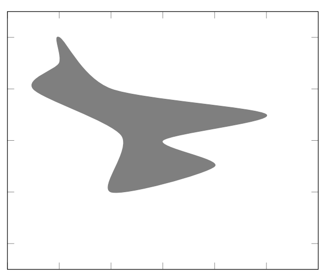


Your friend then claims: A-ha, the intermediate value theorem is wrong! This function never hits the value -1!

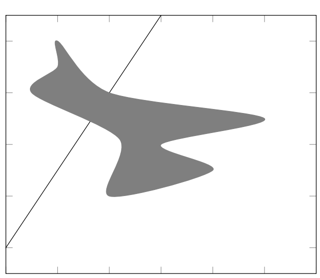
Is your friend correct? Is all of calculus a failure? Explain.

22.2 A fun exercise: Wonky pizza

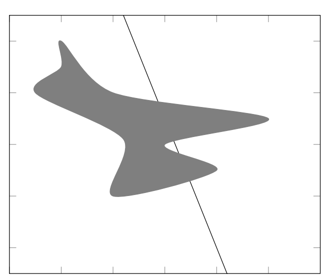
Here is a picture of a wonky-shaped pizza. (And yes, it's gray; not the most tasty-looking thing, is it?)



Your boss wants you to cut this pizza in half, using *one*, linear cut. For example,



and



are two cuts you're allowed to make. Notice that the resulting pizza can have more than just two pieces (as seen on the righthand cut). All that your boss wants is that all the pizza on one side of the cut, has the same area as all the pizza on the other side of the cut.

Exercise 22.2.1. Using the Intermediate Value Theorem, convince yourself that for *any* slope m you choose, you can make a cut of slope m such that you divide the pizza into equal halves (just as your boss requires).

Does the theorem tell you *where* to cut the pizza?

22.3 Extreme Value Theorem

The Extreme Value Theorem says that if a function f is continuous on a *closed, bounded* interval, then f attains a maximum and a minimum there.

Let's see an example before we get to the theorem itself.

Example 22.3.1. Let $f(x) = x^4 - 3x + 9$. You might have no idea what this looks like, but you at least know it's continuous because it's a polynomial. Now, fix two numbers a and b – let's say $a = -5$ and $b = 13$. Intuitively, it probably makes sense to you that there is some maximal value that f attains along the interval $[a, b]$. This value might be at a, it might be attained at b, or it might be attained somewhere inbetween. Likewise, it probably makes sense that f attains a minimum along this interval.

The extreme value theorem doesn't tell you *where* maxima and minima exist. But it does tell you that they *do* exist.

Remark 22.3.2. We've already tackled problems that ask you to find minima and maxima. What the extreme value theorem tells you is that your professors weren't sending you on wild goose chases – along a closed interval, maxima and minima do exist.

Remark 22.3.3. Just to sort things out: The extreme value theorem tells you that you *can* look for maxima and minima because they're out there somewhere. This is for continuous functions. When a function is furthermore differentiable twice (so it has a derivative and a second derivative) then our usual methods from earlier in the semester help us actually find them.

Example 22.3.4. Here are some examples.

(a) For the function $f(x) = x$ along the interval $[2, 7]$, the maximum is attained at $x = 7$. Note that this shows that maxima can happen at the *endpoint* of a closed interval.

(b) This example of f also shows that the function does *not* attain a maximum along the open interval $(2, 7)$. No matter what point x you choose in that open interval, you can find another point x' with $f(x') > f(x)$.

(c) For the function $f(x) = 7-x^2$ along the interval $[-3, 4]$, the maximum is attained at $x = 0$. This example shows that a maximum can happen on the interior of the closed interval.

(d) Consider the function $f(x) = 1/x$. This function does *not* attain a maximum along the closed interval $[-3, 4]$. After all, f just keeps getting larger and larger as x approaches 0 from the right. The issue here is that f is not continuous along this interval – it isn't even defined at $x = 0$.

Here is the extreme value theorem:

Theorem 22.3.5. Fix a function f and two real numbers a and b with $a < b$. If f is a function continuous along the interval $[a, b]$, then there exists some number $c \in [a, b]$ so that for all $x \in [a, b]$, we have

$$f(x) \leq f(c).$$

In other words, f attains a maximum somewhere along the interval $[a, b]$.

22.4 Why do we care about IVT?

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The philosophical take-away is twofold:

(a) The notion of "continuous," which is very formal, guarantees properties that conform to our intuitions. This is evidence that the definition of "continuous function" (as given last lecture) reasonably captures some of our intuition.

(b) Continuity is a *useful* property of a function. If one is handling a function f, and *if f is continuous*, one can sleep well at night knowing f behaves in ways we like.

As a student, you probably do not care too much that a "definition" of a word like continuous has nice properties. It's probably all the same to you because you might as well assume everything is continuous. You are completely justified in thinking so.

But real fissures in science can appear in the details. Imagine a world in which scientists are still feuding over what continuity ought to mean. Worse, imagine a world where scientists cannot agree whether "nice" functions should satisfy the IVT.

(In life, you have probably gotten into frustrating discussions that go nowhere. Many times, this is simply because you and your discussion partner are arguing over things that are not defined, or things for which you genuinely have different definitions. Having a set starting point – a set definition – allows you to progress the discussion together.)

What is rarely taught in high school is that the evolution and progress of mathematics depends on creativity and on precision both – on the artistic and the meticulous. Just as the most beautiful writing is not the most flowery, but the most true, it turns out that these definitions have also given incredible beauty in math with their ability to capture absurd amounts of intuition through incredibly minimal assumptions.

22.5 Back to limits: The derivative of the absolute value function

I would like now to give an example of a limit that illustrates a feature of the derivative we haven't quite talked about. Some functions might *not* have a derivative.

Exercise 22.5.1. Let $f(x) = |x|$. Another definition for the absolute value function is:

$$|x| := \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0. \end{cases}$$

(a) Write out the difference quotient of f at $x = 0$. (Remember: This is a function of the variable h, undefined where $h = 0$.)

(b) What is $\lim_{h\to 0^+}$ of the difference quotient you wrote?

(c) What is $\lim_{h\to 0^-}$ of the difference quotient you wrote?

(d) Does your difference quotient have a limit at $h = 0$?

(e) Does $|x|$ have a derivative at $x = 0$?

22.6 Bonus Material: Continuity on a closed interval, and Intermediate value theorem on a closed interval

Recall that a *closed* interval is an interval of the form

$$[a, b]$$

with $a < b$. For example, $[2, 7]$ is the interval of all numbers between 2 and 7, *including* 2 and 7.

An *open* interval is an interval of the form

$$(a, b)$$

with $a < b$. For example, $(2, 7)$ is the interval of all numbers between 2 and 7, *not including* 2 and 7.

If a function $f(x)$ is defined only on a closed interval $[a, b]$, it's not obvious what we mean for f to be continuous—mainly because we can only define a one-sided limit (and not a limit) at a and b. But we take what we can get:

Definition 22.6.1. If a function $f(x)$ is defined only on a closed interval $[a, b]$, we say that f is *continuous at a* if

1. The righthand limit $\lim_{x \to a^+} f(x)$ exists, and
2. $\lim_{x \to a^+} f(x) = f(a)$.

Likewise, we say that f is *continuous at b* if

1. The lefthand limit $\lim_{x \to b^-} f(x)$ exists, and
2. $\lim_{x \to b^-} f(x) = f(b)$.

We say that f is continuous if it is continuous at every point of $[a, b]$. [5]

Theorem 22.6.2. The intermediate value theorem holds for continuous functions defined on a closed interval.

[5] Note that for any element c inside of (a, b)—that is, for any c with $a < c < b$—we know what it means for $f(x)$ to be continuous at c, because we know how to define the limit of f at c.

22.7 For next time

For next time, I expect you to be able to use the puncture law to compute limits of rational functions. For example, you should be able to compute the following limits:

1. $\lim_{x\to 0} \frac{x^3+3x^2}{x^2}$.
2. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.
3. $\lim_{x\to -2} \frac{x^2-4}{x^2+x-2}$.

Also in preparation for next lecture, you should be able to answer the following questions:

(a) What are the *three conditions* you need to check to see whether a function $f(x)$ is continuous at a?

(b) Why is $f(x) = 3/x$ not continuous at zero?

(c) Why is $f(x) = 8/x$ continuous at 5?