22.8 Lab exercises: Limits, continuity, and applications

22.8.1 Intermediate value theorem

Exercise 22.8.1. The following are claims about certain functions. Justify them using the intermediate value theorem, making sure to specify what your choices of f, a, b, and N are. Make sure that you justify why the intermediate value theorem can be used by stating $f(a)$ and $f(b)$.

Secretly, the difficulty of these problems are not just about applying the IVT, but also about knowing the behavior of some of these functions.

(a) There exists some x for which $\sin(x) = 0.2$.

(b) There exists some x for which 2^x equals 11.

(c) There exists some x for which $\log_3(x)$ equals 1.1.

(d) There exists some x for which $\tan(x)$ equals 0.1.

(e) There exists some number whose cube is 13.

(f) There exists some number who squares to π.

Isn't it cool that the intermediate value theorem shows that each of the above statements is true?

Exercise 22.8.2. Your friend is making a series of logical deductions using the Intermediate Value Theorem. For each statement below, identify whether the logical deductions are correct or not. When the deduction is incorrect, explain why the Intermediate Value Theorem is not being used correctly.

(a) Let $f(x) = x^2$. Then $f(1) = 1$ and $f(2) = 4$, so there is some number whose square is 3.

(b) Let $f(x) = x^2$. Then $f(1) = 1$ and $f(2) = 4$, so there is some number whose square is 9.

(c) Let $f(x) = \sin(x)$. Then $f(0) = 0$ and $f(\pi) = 0$, so there is some number x between 0 and π for which $f(x) = 1$.

(d) Let $f(x) = \sin(x)$. Then $f(0) = 0$ and $f(\pi/2) = 1$, so there is some number x between 0 and $\pi/2$ for which $f(x) = 0.2$.

(e) Let $f(x) = 1/x$. Then $f(-1) = -1$ and $f(1) = 1$, so there is some number x between -1 and 1 for which $f(x) = 0$.

(f) Let $f(x) = \frac{1}{x-2}$. Then $f(1) = -1$ and $f(3) = 1$, so there is some number x between 1 and 3 for which $f(x) = 0$.

(g) Let $f(x) = \sqrt{x}$. Then $f(9) = 3$ and $f(16) = 4$, so there must be some number between 9 and 16 whose square root is π.

(h) Let $f(x) = \sqrt{x}$. Then $f(9) = 3$ and $f(16) = 4$, so there must be some number between 9 and 16 whose square root is e.

22.8.2 Computing limits

Exercise 22.8.3. Using the puncture law, compute the following limits. Explain also why you cannot just "plug in" the right value of x into the fraction to find the limit.

(a) $\lim_{x\to 5} \frac{x^2-25}{(x-5)(x-3)}$

(b) $\lim_{x\to 5} \frac{x^2-25}{x^2-7x+10}$

(c) $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$. (Name also the function f and the number a for which you have just computed $f'(a)$.)

(d) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

(e) $\lim_{t\to 2} \frac{\sqrt{t}-2}{t-4}$

Exercise 22.8.4. Consider the function

$$f(x) = \begin{cases} x^2 - 4 & x < -1 \\ 3x & -1 < x < 0 \\ \sin(x) & 0 < x < \pi \\ x^2 & \pi < x < 2\pi \\ \cos(x) & 2\pi < x. \end{cases}$$

For every value of a below, compute $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$. Then, state whether $\lim_{x\to a} f(x)$ exists.

(a) $a = -1$.

(b) $a = 0$.

(c) $a = \pi$.

(d) $a = 2\pi$.

22.8.3 Challenge problems

Exercise 22.8.5. Can you come up with a function that ...

(a) has a derivative everywhere but that is not continuous everywhere? Why or why not?

(b) is not continuous everywhere but has a derivative everywhere? Why or why not?

(c) is continuous everywhere but lacks a derivative somewhere? Why or why not?

(d) has a derivative everywhere but lacks a second derivative somewhere? Why or why not?