Lecture 21

Limits, one-sided limits, and continuity

Today, we're going to start talking about the idea of *limits*. Calculus so far has been about two ideas: derivatives and integrals. Both of these are defined using limits.

Remark 21.0.1 (Some motivation.)**.** We have defined the derivative to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

As you know, the righthand side is the value that the difference quotient *approaches* as h goes to zero. So, limits have already made an appearance in our lives, and we already have an intuition of what limits are supposed to be. We'll make these intuitions more precise starting this lecture.

Warning 21.0.2. In real life, the word "limit" can have many meanings. For example, a speed limit tells you that you shouldn't go above a certain speed. Sometimes, a limit is also an "extreme" that you can't get past. Neither of these meanings is what limit means in calculus class. In calculus class, a limit is something we *approach.*

21.1 The basic idea

Let's say that we have a function called $f(x)$. Then we can ask whether $f(x)$ approaches a certain (output) value as x approaches a certain (input) value.

Notation 21.1.1 (Informal.)**.** Let f be a function. The notation

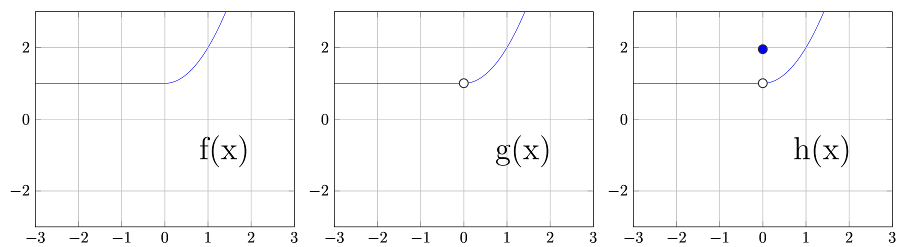
$$\lim_{x \to a} f(x)$$

stands for the number that $f(x)$ approaches as x approaches a.

Warning 21.1.2. Limits might not exist at particular values of a—for example, maybe f doesn't want to approach a single value as x approaches a. We will see examples of this later today.

21.2 Open and solid dots

Let's first review what we mean when we draw certain things. Below are graphs of three functions, f, g and h.



The difference between the three graphs, of course, are the presence of certain dots. Remember that when we draw pictures in math, an "open dot" (e.g., the white dots in the middle picture and the righthand picture) signifies a missing point. In other words, the open dot indicates a point that is *not* on the graph. So the point $(0, 1)$ is *not* a point on the graph of g. In other words, $g(0)$ does not equal 1. In fact, g does not take on any value at $x = 0$. So we say that g is *not defined* at $x = 0$.

Consider, on the other hand, the graph of $h(x)$. The open dot tells us that $h(0)$ does not equal 1. But there is a solid dot at $(0, 2)$. This tells us that $h(0) = 2$. Solid dots indicate that the dot *is* part of the graph.

In contrast, $f(0) = 1$.

Remark 21.2.1. Oftentimes, we draw open and solid dots when the function does something visually unexpected or funny; for example, in the graph of $h(x)$ and $g(x)$, it looks like both functions want to take on the value 1 at $x = 0$. The dots indicate that this is not so. You can see why dots might come up in typical limit problems in calculus class.

21.3 Some simple limits, and basic lessons

So let's practice.

Exercise 21.3.1. Using the graph of f above, tell me the following numbers:

(a) $\lim_{x\to 0} f(x)$.

(b) $\lim_{x\to 1} f(x)$.

(c) $\lim_{x\to -2} f(x)$.

Now tell me the above limits by replacing f with g. And then do it for h.

Possible solutions. The solutions to the exercise can be "read off" from the graph. As x approaches 0, we see that the blue graph approaches the height of 1. So the first limit is 1. For the next part: As x approaches 1, we see the blue curve wants to (and does) attain a height of 2. Finally, as x approaches -2, we see that the blue graph wants to maintain (and does) attain a height of 1. So

(a) $\lim_{x\to 0} f(x) = 1$.

(b) $\lim_{x\to 1} f(x) = 2$.

(c) $\lim_{x\to -2} f(x) = 1$.

So what about for g and h? In fact, the answers for g and h are *identical* to the answers for f. In other words, f, g, and h have the exact same limits! (Even though, clearly, they are different functions—at $x = 0$, g is not defined, while h is defined there and takes a value of 2 there.) □

So we learn some lessons:

A function does not need to be defined at $x = a$ *to have a limit* $\lim_{x\to a} f(x)$.

(As we saw for g at $x = 0$.)

The value of the function $f(a)$ *does not need to equal the limit* $\lim_{x\to a} f(x)$.

(As we saw for h at $x = 0$.)

21.4 One-sided limits

Sometimes, a function approaches a value from the right; sometimes, the function approaches a value from the left. These values might be different!

Notation 21.4.1 (One-sided limits, informally)**.** If $f(x)$ wants to converge to a value as x approaches a from the right, we call this value the *righthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x \to a^+} f(x).$$

(Note the plus sign on the a.)

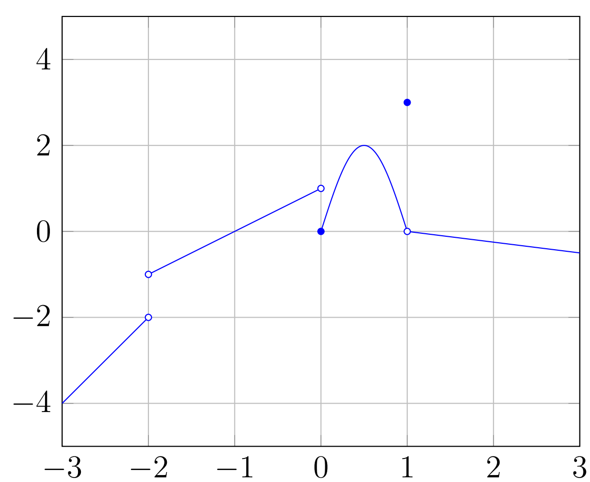
If $f(x)$ wants to converge to a value as x approaches a from the left, we call this value the *lefthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x \to a^-} f(x).$$

(Note the *minus* sign on the a.)

A lefthand limit or a righthand limit is called a *one-sided limit*.

Exercise 21.4.2. Below is the graph of a function $f(x)$.



Based on the graph, give your best guest for the following one-sided limits.

(a) $\lim_{x \to -2^-} f(x)$.

(b) $\lim_{x \to -2^+} f(x)$.

(c) $\lim_{x \to 1^+} f(x)$.

(d) $\lim_{x\to 1^-} f(x)$.

(e) What is $f(1)$?

(f) What is $f(0)$?

(g) What is $f(-2)$?

Here are solutions:

(a) $\lim_{x\to -2^-} f(x) = -2$. This one-sided limit asks what value f approaches as x approaches -2 from the left.

(b) $\lim_{x\to -2^+=-1} f(x)$. This one-sided limit asks what value f approaches as x approaches -2 from the right.

(c) $\lim_{x\to 1^+} f(x) = 0$.

(d) $\lim_{x\to 1^-} f(x) = 0$.

(e) $f(1) = 3$.

(f) $f(0) = 0$.

(g) This is a trick question. f is not defined at -2.

Example 21.4.3. Consider the function

$$f(x) = \begin{cases} 0 & x > 0 \text{ and } x \text{ is irrational} \\ 1 & x > 0 \text{ and } x \text{ is rational} \\ 13 & x < 0. \end{cases}$$

Then the one-sided limit $\lim_{x\to 0^+} f(x)$ does not exist. (This is a subtle point – the reason is that as x approaches 0 from the right, the value of f oscillates between 0 and 1 without approaching any value.) On the other hand, the one-sided limit $\lim_{x\to 0^-} f(x)$ does exist – it is 13.

Note that f is not defined at 0.

Example 21.4.4. Consider the function

$$f(x) = \begin{cases} 0 & x < 0 \text{ contains at least a single instance of 1 in its decimal expansion} \\ 1 & x < 0 \text{ and } x \text{ contains no instances of 1 in its decimal expansion} \\ e & x > 0. \end{cases}$$

Then the one-sided limit $\lim_{x\to 0^-} f(x)$ does not exist. (The reason is the same as the previous example. As x approaches 0 from the left, the value of f oscillates between 0 and 1 without approaching any value.) On the other hand, the one-sided limit $\lim_{x\to 0^+} f(x)$ does exist – it is e.

Note that again, f is not defined at 0.

21.5 Using one-sided limits

Here is our first theorem about limits. A *theorem* is a true statement that requires an involved proof, and the true statement is so useful that we should[1] know it for future use.

Theorem 21.5.1. The following statements are equivalent:

1. $f(x)$ has a limit at a.
2. Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, and the one-sided limits agree.

Moreover, in this situation, we can conclude that

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Remark 21.5.2. The term "equivalent" has a precise meaning here. It means that "if the first statement is true, then the second statement true," *and* that "if the second statement is true, then the first statement is true."

In other words, if f has a limit at a, then it has both one-sided limits there, and they agree. Conversely, if f has both one-sided limits at a and they agree, then f has a limit at a.

Example 21.5.3. Somebody tells you the following information:

$$\lim_{x\to 1^+} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 10.$$

Then you know that $\lim_{x\to 1} f(x)$ does not exist, because the two one-sided limits do not agree.

Example 21.5.4. Somebody tells you the following information:

$$\lim_{x\to 2^+} f(x) = 10 \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = 10.$$

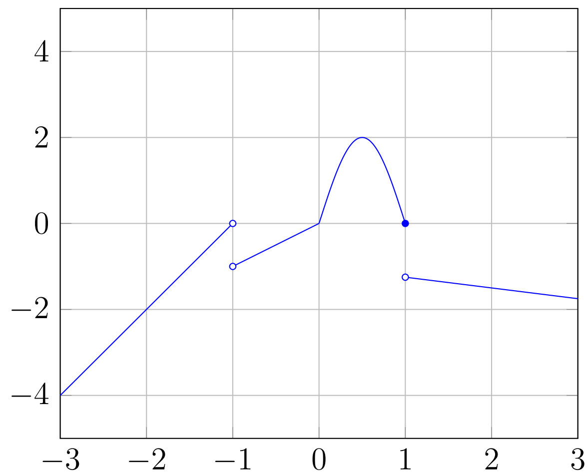
Then you know that $f(x)$ *does* have a limit at 2, because the two one-sided limits agree (that is, they have the same value). Moreover, you can conclude that

$$\lim_{x\to 2} f(x) = 10.$$

[1]That means you'll be tested on it!

21.6 Some exercises

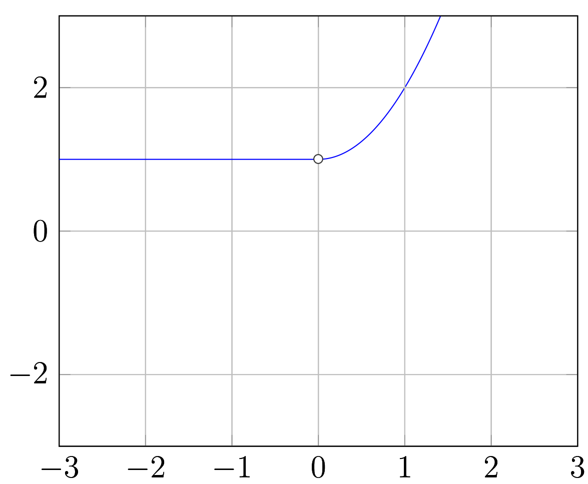
Exercise 21.6.1. At which points does the following function *not* have a limit?

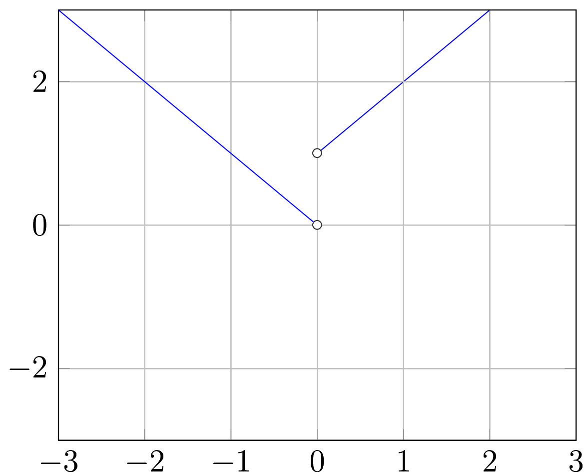


Exercise 21.6.2. For the functions below, determine whether the limit

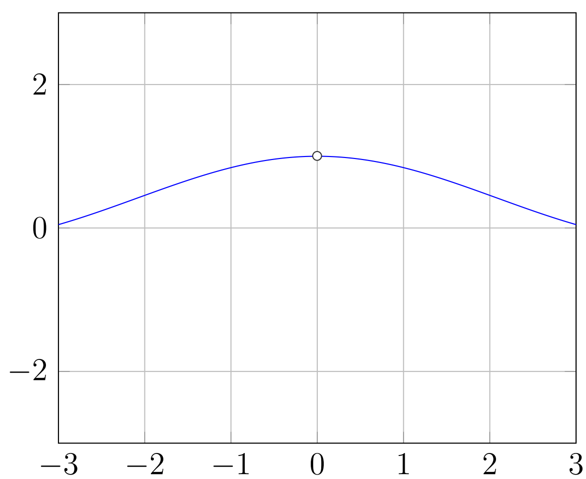
$$\lim_{x \to 0} f(x)$$

exists; and if so, say what the limit is.

(a) 



(b)



(c)

Exercise 21.6.3. Draw an example of a graph of f satisfying the following properies:

1. $f(1) = \lim_{x\to 1} f(x)$.

2. $f(2) \neq \lim_{x\to 2} f(x)$.

3. f is not defined at $x = 3$ but $\lim_{x\to 3} f(x)$ exists.

4. $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4^-} f(x)$ exist but $\lim_{x\to 4} f(x)$ does not.

21.7 Continuity

Now, you probably get the feeling that the kinds of graphs we've drawn today are different from the kinds of graphs we've drawn earlier in this course. (Earlier in this course, we didn't need open dots or closed dots, for example.) It turns out there's a name for the kinds of phenomena that we are seeing today:

Definition 21.7.1. A function f is called *continuous at a* if

1. $f(a)$ is defined,

2. $\lim_{x\to a} f(x)$ exists, and

3. $\lim_{x\to a} f(x) = f(a)$.

If a function is continuous at a, you can think of f as looking "nice" near a. Here's one intuition: If a function is continuous at a, it means that you can draw the graph of f near a without ever lifting your pencil from the sheet of paper (because the graph won't have a "jump"). If f is continuous everywhere (i.e., at every a) then you can draw the entire graph of f without ever lifting your pencil.

As it turns out, most of the functions that we've studied in this class have been continuous everywhere they've been defined; this is why we can draw their graphs without having to "jump" or lift a pencil.

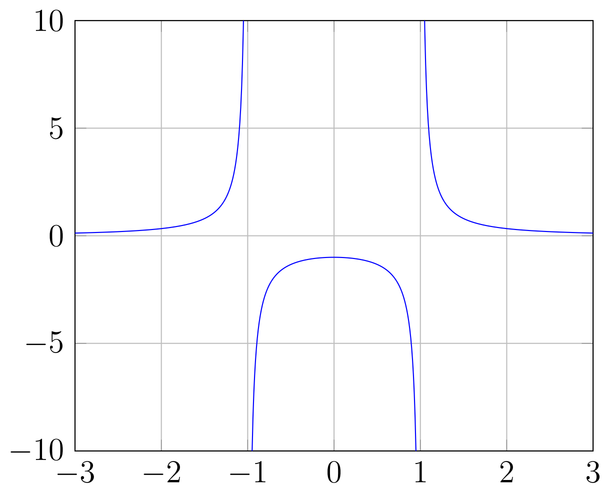
Definition 21.7.2. A function $f(x)$ is called *continuous* if it is continuous at every point that $f(x)$ is defined.

Intuition: "*A continuous function is one for which you can draw the graph of the function without ever having to lift your pencil from the paper.*"

Warning 21.7.3. This intuition fails in small ways. For example, suppose that

$$f(x) = \frac{1}{(x+1)(x-1)}.$$

Here is the graph of $f(x)$:



You can see that f is not defined at $x = 1$ and $x = -1$. So there is no way that you can draw the whole graph without lifting your pencil. But f is still a continuous function, because the value of f agrees with the limit of f at every point f is defined.

Regardless, "never have to lift your pencil" is a useful way to think about what continuity looks like. This agrees with another intuition: A continuous function has no "sudden jumps."

Example 21.7.4. As it turns out, almost every function with a "formula" that you know is continuous. Here is a list of some examples of continuous functions:

1. $f(x) = 10$ (and all other constant functions)

2. $f(x) = x$ (and all other linear functions)

3. $f(x) = 3x^3 + 4x^2 + 9$ (and all other polynomials—you can actually prove this based on the basic limit laws we will learn next time)

4. $f(x) = \frac{3x^2+1}{x-3}$ (and all other functions that are quotients of polynomials)

5. $f(x) = |x|$ (I bet you can prove this function is continuous!)

6. $f(x) = \sin(x)$ (and all other trig functions)

7. $f(x) = \sqrt{x}$

8. $f(x) = x^p$, for any real number p, and when x is non-negative. (You should be familiar with the special cases when p is a negative integer like $p = -1$ or $p = -2$, and when p is a fraction like $p = 1/3$ or $p = 2/3$.)

9. $f(x) = e^x$

10. $f(x) = \ln(x)$

The continuity of the last five examples require some proofs that we won't go over in this class.

From now on, you may use—and are expected to know—that all the functions above are continuous.

21.8 Limits for functions that aren't presented visually

Now that you know that most functions with reasonable formulas are continuous, you can begin to answer questions about limits for functions that aren't presented visually (but presented using formulas).

Exercise 21.8.1. Below are some functions $q(h)$. Each function q is defined everywhere except at $h = 0$. For each, determine whether the limit

$$\lim_{h \to 0} q(h)$$

exists; and if so, say what the limit is.

(a) $q(h) = \begin{cases} h^2 & h \neq 0 \end{cases}$

(b) $q(h) = \begin{cases} \sin(h) & h > 0 \\ \cos(h) & h < 0 \end{cases}$

(c) $q(h) = \begin{cases} 1 & h \text{ is a rational number and } h \neq 0 \\ 0 & h \text{ is an irrational number} \end{cases}$

Remark 21.8.2. Recall that a rational number is a number that can be expressed as a fraction—things like $-2/7$, or 13, or $5/6$. An irrational number is a real number that is not a fraction. For example, $\sqrt{2}$ or π.)

Remark 21.8.3. Recall that a function is called *piecewise defined* when it is defined in the following format:

$$q(h) = \begin{cases} \text{blah blah} & \text{some condition on } h \\ \text{blahbitty blah} & \text{some other condition on } h \\ \text{Rob Loblaw} & \text{perhaps another condition on } h \end{cases}$$

We tend to define functions using the above format when it's not easy to define the function in one fell swoop. For example, the function (c) above means that $q(h)$ equals 1 when h is a non-zero rational number, and equals 0 when h is an irrational number.

21.9 Real-world examples of discontinuity; modeling

Some people will claim to you that there are real-world examples of discontinuous functions. Like so many things in life, this really depends on how you decide to measure the inputs and outputs of functions.

For example, people like to cite population (with respect to time) as discontinuous because population is only measured in whole numbers. But it is useful to model population using continuous functions that pretend we can have a real-number worth of humans. And, as you know, intense political and philosophical debates can emerge when one discusses what constitutes a human being at all!

And while many functions seems to have "sudden jumps" (such as the transition from static to kinetic friction), it is difficult to ascertain whether such a jump is actually a continuous rise happening at incredibly small scales.

Finally, while we model space and time using real numbers, a lot of what seems continuous at a macroscopic scale is quantized and jumps between discrete values at a miniscule scale. So there is a case to be made that certain functions in life are actually very much discontinuous.

All this is to say that whether functions you care about in real life are discontinuous or not is often dependent on the scales and details you are willing to tolerate in your model. It becomes more a discussion of implementation and model-choosing, rather than a discussion of what the underlying reality is.

With this in mind, let me say that there are indeed functions in the real world that seem to "suddenly jump" at the scales we care about. The amount of friction car tires experience, for example, can suddenly transition between kinetic and static friction.