21.10 Lab exercises on limits and limit laws

In today's lab, I want you to work through this worksheet as far as you can get. We will be learning some *limit laws* in this lab through doing.

In each part of the lab, you will be told a basic fact about limits that you may use from now on in this course. These basic facts allow you to compute limits *without* invoking the fact that "functions with formulas tend to be continuous."

Based on each new basic fact, I want you to complete new exercises.

21.10.1 Limits of constants

If $f(x)$ is a constant function[2] with value C, then

$$\lim_{x\to a} f(x) = C$$

regardless of a.

Exercise 21.10.1. Compute the following limits:

(a) $\lim_{x\to 3} 3$

(b) $\lim_{x\to 1} 3$

(c) $\lim_{x\to -1} 3$

(d) $\lim_{x\to -1^+} 3$

(e) $\lim_{x\to -1^-} 3$

(f) $\lim_{x\to -1} \pi$

(g) $\lim_{x\to -1^+} e$

(h) $\lim_{x\to -1^-} \sqrt{2}$

[2]This means $f(x) = C$ for some number C. Put another way, the graph of $f(x)$ is just a flat, horizontal line.

21.10.2 Limits of x.

For the function $f(x) = x$, we have that

$$\lim_{x \to a} f(x) = a.$$

(I encourage you to graph the function $f(x) = x$; then this law will seem "obvious" to you.)

Remark 21.10.2. The first two laws are hopefully not too bewildering; the notation is confusing, but these are meant to be among the simplest examples. I state these just to get our feet wet; it's the next few laws that will really get us going.

Exercise 21.10.3. Compute the following limits:

(a) $\lim_{x \to 3} x$

(b) $\lim_{x \to 1} x$

(c) $\lim_{x \to -1^+} x$

(d) $\lim_{x \to -1^-} x$

(e) $\lim_{x \to e^-} x$

(f) $\lim_{x \to e^+} x$

(g) $\lim_{x \to e} x$

21.10.3 Limits scale.

If a limit already exists, then the limit of the scaled function is the scaled limit of the function. More precisely: *If* $\lim_{x \to a} f(x)$ *already exists*, then for any number k, we have the following:

$$\lim_{x \to a} (k \cdot f(x)) = k \cdot \left(\lim_{x \to a} f(x) \right)$$

Exercise 21.10.4. Compute the following limits:

(a) $\lim_{x \to 3} 7x$

(b) $\lim_{x \to 1} 8x$

(c) $\lim_{x \to -1^+} \pi x$

(d) $\lim_{x\to -1^-} ex$

(e) $\lim_{x\to e^-} -\sqrt{3}x$

(f) $\lim_{x\to e^+} \sqrt{2}x$

(g) $\lim_{x\to e} -2x$

21.10.4 Limits add.

If the limits already exist, then the limit of the sum exists; moreover, the sum of the limits is the limit of the sum.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) + g(x))$ exists, and

$$\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

Exercise 21.10.5. Compute the following limits:

(a) $\lim_{x\to 3} 7x + 1$

(b) $\lim_{x\to 1} 8x - 2$

(c) $\lim_{x\to -1^+} \pi x - \pi$

(d) $\lim_{x\to -1^-} ex + 3$

(e) $\lim_{x\to e^-} \sqrt{2} - \sqrt{3}x$

21.10.5 Limits multiply.

If limits already exist, then the limit of the product exists; moreover, the product of the limits is the limit of the product.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) \cdot g(x))$ exists, and

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

Exercise 21.10.6. Compute the following limits:

(a) $\lim_{x\to 3}(7x + 1)(3x - 1)$

(b) $\lim_{x\to 1} x(8x - 2)$

(c) $\lim_{x\to -1^+} x \cdot x$

(d) $\lim_{x\to -1^-} x^2$

(e) $\lim_{x\to e^-} x^3$

(f) $\lim_{x\to 1^-} x^3 + 3x^2 - 1$

(g) $\lim_{x\to 1^+} 4x^3 + 3x^2 - 1$

(h) $\lim_{x\to -1^+}(4x^3 + 3x^2 - 1)(x - 1)$

21.10.6 Limits divide.

If limits already exist, then the limit of the quotient exists; moreover, the quotient of the limits is the limit of the quotient (provided the denominator is not zero).

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x)/g(x))$ exists, and

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

so long as $\lim_{x\to a} g(x) \neq 0$.

Exercise 21.10.7. Compute the following limits:

(a) $\lim_{x\to 3} \frac{3x-1}{9+x}$

(b) $\lim_{x\to 1} \frac{x^2}{x+1}$

(c) $\lim_{x\to -1^+} \frac{9x-3}{(x-1)^2}$

(d) $\lim_{x\to -1^-} \frac{x^2+8x}{9-x}$

(e) $\lim_{x\to e^-} \frac{x-1}{x+1}$

Remark 21.10.8. The above limit laws have three parts: (i) The *criterion* that certain limits already exist, (ii) The *guarantee* that another limit exists, and (iii) The *formula* of how to compute that other limit.

I wrote all the formulas in such a way that the righthand side of the formula consists of the limits given to exist (by the criterion); the lefthand side is the limit that we are then guaranteed to exist.

Remark 21.10.9. It's important to note that, for every law, the limits are taken at the same point. That is, every limit in sight is taken as x approaches a single number a. So for example, even if I know that $\lim_{x\to a} f(x)$ exists, and that $\lim_{x\to b} g(x)$ exists, I don't know anything about the limits of $f(x) + g(x)$ unless $a = b$. (In which case, I know that a limit exists as $x \to a$.)

Exercise 21.10.10. Using some of the facts above, show that **limits subtract**.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a} (f(x) - g(x))$. Moreover,

$$\lim_{x\to a} (f(x) - g(x)) = \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} g(x)\right)$$

(Hint: Use the fact that limits scale, taking your scaling constant to be $k = -1$, and use the fact that limits add.)

Exercise 21.10.11. Use the limit laws to compute

$$\lim_{x\to 1} \left(\frac{x^2+3}{x}\right).$$

What goes wrong when you try to compute the limit as $x \to 0$?

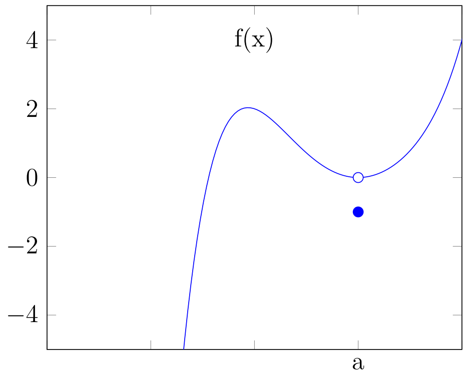
21.11 Puncture law

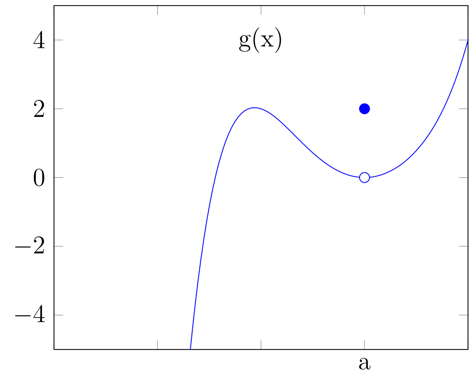
Let $f(x)$ and $g(x)$ be two functions. Suppose that the two function are equal at all points nearby a (but not necessarily at a itself). Then $f(x)$ has a limit at a if $g(x)$ does, and likewise, $g(x)$ has a limit at a if $f(x)$ does. Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Warning 21.11.1. Many calculus textbooks do *not* talk about a "puncture law." In my opinion, this is a bit ludicrous, because about half of the algebraic "tricks" we have to compute limits are dependent on it. I must admit that I made up the term "puncture law," so you may find your peers outside of your class being confused if you use this law.

Example 21.11.2 (A graphical example)**.** On the left is a graph of $f(x)$, and on the right is a graph of $g(x)$.





Note that the value of $f(x)$ and $g(x)$ are different at a (the filled dots are at different heights).[3] But $f(x)$ and $g(x)$ are otherwise identical, so they have the same limit at a. This "obvious" fact is called the puncture law.

Exercise 21.11.3. Compute $\lim_{x\to 0} \frac{x^2}{x}$.

A solution. Let

$$f(x) = \frac{x^2}{x} \qquad \text{and} \qquad g(x) = x.$$

Note that $f(x)$ is not defined at $x = 0$, but is equal to $g(x)$ for all other values of x. Thus, the puncture law tells us that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x). \tag{21.11.1}$$

Of course, you know what the righthand side is (by plugging in what $g(x)$ is):

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0. \tag{21.11.2}$$

So, putting (21.11.1) and (21.11.2) together, we see that

$$\lim_{x\to 0} f(x) = 0.$$

In other words (by plugging in the definition of $f(x)$) we find:

$$\lim_{x\to 0} \frac{x^2}{x} = 0.$$

Note that this is an example where the quotient law wouldn't help you, because the limit of the denominator equals zero! □

[3] Let me remind you—as I mentioned in class—that the white dot means that the function does *not* take the value of the white dot there. The black dot indicates the value of the function. Often, we write a white dot where it looks like a function wants to take a value, but does not.

Exercise 21.11.4. Find the limit

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2}.$$

Possible solution. Note that the function we have is *undefined* when $x = 2$ (because we can't divide by $x - 2$ when $x = 2$). But, we know the following:

$$\frac{(x+1)(x-2)}{x-2} = x+1 \qquad \textit{so long as } x \neq 1.$$

In other words, the two functions

$$\frac{(x+1)(x-2)}{x-2} \qquad \text{and} \qquad x+1$$

are *equal* away from $x = 1$. Thus, the puncture law tells us

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2}(x+1).$$

Now, let's just compute the righthand side:

$$\begin{aligned} \lim_{x\to 2}(x+1) &= \lim_{x\to 2} x + \lim_{x\to 2} 1 \qquad (21.11.3) \\ &= 2+1 \\ &= 3. \end{aligned}$$

(We used the addition law in line (21.11.3).) Putting everything together, we conclude:

$$\frac{(x+1)(x-2)}{x-2} = 3.$$

We're done, but let me streamline everything to show you what you might be able to write on a test:

$$\begin{aligned} \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law} \\ &= \lim_{x\to 2} x + \lim_{x\to 2} 1 && \text{by the addition law} \\ &= 2+1 \\ &= 3. \end{aligned}$$

Another solution you might write on a test is:

$$\begin{aligned}
\lim_{x\to 2}\frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law}\\
&= 2+1 && \text{because polynomial functions are continuous}\\
&= 3.
\end{aligned}$$

□

Exercise 21.11.5. Compute [4]

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9}.$$

Possible solution. This looks very complicated; to use the puncture law, we'd like to find some other function that is equal to $\frac{x^2-2x-3}{x^2-9}$ away from 3. The trick I want you to learn here is that you can *cancel* $(x-3)$ in the top and bottom. This may seem very confusing, because $(x-3)$ doesn't appear anywhere in the function as it's presented. But you'll see that it *does* appear if you factor.

Pro tip. Why do you want to try to cancel $x-3$? It's because we should feel that a term of the form "$x-3$" is what's causing the denominator to equal zero at $x=3$. So it's natural to try and see if, indeed, a factor of $(x-3)$ can pop up in the denominator. More generally, for rational functions, *if you are computing a limit as x approaches a, it is natural to try to find $(x-a)$ as a factor of the top and bottom.*

Warning. If you don't know how to divide or factor polynomials, you should make sure to Google and practice—sometimes we'll need to know how to divide polynomials using long division, or how to factor polynomials through other tricks.

In fact, we can factor both the top and the bottom:

$$\frac{x^2-2x-3}{x^2-9} = \frac{(x-3)(x+1)}{(x-3)(x+3)}.$$

And we see that we can cancel the $(x-3)$ terms! So, when x does not equal 3, our function $\frac{x^2-2x-3}{x^2-9}$ is equal to

$$\frac{x+1}{x+3}. \tag{21.11.4}$$

By the puncture law, we thus conclude the following:

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9} = \lim_{x\to 3}\frac{x+1}{x+3}.$$

[4] Note that the quotient law doesn't help here, because the limit of the denominator equals zero.

And, as we saw, *any rational function is continuous where it is defined.* The rational function in (21.11.4) is defined at $x = 3$, so—by the definition of continuity—we can compute the limit simply by plugging 3 into x:

$$\lim_{x \to 3} \frac{x+1}{x+3} = \frac{3+1}{3+3} = \frac{4}{6} = \frac{2}{3}.$$

Putting everything together, we conclude

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} = 2/3.$$

□

Exercise 21.11.6. Compute the following limits:

1. $\lim_{x \to 0} \frac{x^3+3x^2}{x^2}$.
2. $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$.
3. $\lim_{x \to -2} \frac{x^2-4}{x^2+x-2}$.