Lecture 19

Practice problems

19.1 Conceptual questions

Exercise 19.1.1. (a) Give the definition of $\int_a^b f(t)\,dt$ that we have used in this class.

(b) State the fundamental theorem of calculus, as we have learned it in this class.

Exercise 19.1.2. When computing the indefinite integral, we write

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$.

Tell me why we write "$+C$."

19.2 Taking antiderivatives

Exercise 19.2.1. Compute the following indefinite integrals. Many are taken from Guichard's textbook.

(a) $\int 25 - x^2\,dx$

(b) $\int (1-t)^9\,dt$

(c) $\int (x^2+1)^2\,dx$

(d) $\int x(x^2+1)^{100}\,dx$

(e) $\int \frac{1}{(1-5t)^{1/3}}\,dt$

(f) $\int \sin^3 x \cos x \, dx$

(g) $\int x\sqrt{100 - x^2} \, dx$

(h) $\int \frac{x^2}{\sqrt{1-x^3}} \, dx$

(i) $\int \cos(\pi t) \cos(\sin(\pi t)) \, dt$

(j) $\int \frac{\sin x}{\cos^3 x} \, dx$

(k) $\int \tan x \, dx$

Exercise 19.2.2. Compute the following indefinite integrals.

(a) $\int (3 - 2t)^8 \, dt$.

(b) $\int \tan(x) \, dx$.

(c) $\int \frac{x-1}{3x^2-6x} \, dx$

(d) $\int 7e^x \cos(e^x) \, dx$

(e) $\int \frac{1}{1+x^2} \, dx$

(f) $\int \frac{3}{1+x^2} \, dx$

(g) $\int x^3 + 2x - 9 \, dx$

19.3 Computing integrals

Exercise 19.3.1. Evaluate

$$\int_1^4 \frac{3}{x^2} \, dx$$

Exercise 19.3.2. Evaluate

$$\int_0^\pi \sin^5(3x) \cos(3x) \, dx$$

Exercise 19.3.3. Evaluate

$$\int_1^{e^2} \frac{1}{x} \, dx$$

Exercise 19.3.4. Evaluate

$$\int_1^8 \frac{3x^2 + 2}{\sqrt{x}} \, dx$$

Exercise 19.3.5. Evaluate

$$\int_0^{2\pi} 8\cos(x)\,dx$$

Exercise 19.3.6. Evaluate

$$\int_1^8 2x + 10\,dx$$

19.4 Areas between curves

Exercise 19.4.1. Find the area between the graphs of $x^2 + 2x - 10$ and $4x - 7$.

Exercise 19.4.2. Find the area between the three curves $y = x$ and $y = 7x$ and $x = 1$. (You may want to draw a picture.)

Exercise 19.4.3. Find the area between the graphs of x^3 and x^2.

19.5 Word problems

Exercise 19.5.1. A particle moves with velocity function $v(t) = -t^2 + 3x - 2$. Find the displacement (the signed distance between the starting and ending point) of the particle over the time interval $[-2, 3]$.

Exercise 19.5.2. An epidemiologist models that by day t of a pandemic, her county will have accumulated

$$\int_0^t 10e^{5x}\,dx$$

infections total. According to this, how many new infections per day will her county be seeing at the moment $t = 5$? (Hint: Fundamental Theorem.)

Exercise 19.5.3. It is known that (in the absence of air resistance) if you drop an object at time $t = 0$ seconds, the object's speed at time t is given by

$$v(t) = 32t$$

in feet per second. You drop a coin off a bridge, and you hear the coin hit the water after 10 seconds. Assuming no air resistance, how far above the water was the coin when you dropped it?

Exercise 19.5.4. Your friend is studying an ellipse given by the equation

$$(2x-3)^2 + y^2 = 8^2.$$

Write an integral, or an expression involving an integral, that computes the area of the portion of this ellipse above the x-axis.

Exercise 19.5.5. Your friend is studying a circle given by the equation

$$(x-2)^2 + y^2 = 4$$

Write an integral, or an expression involving an integral, that computes the area of the portion of this circle above the x-axis.

Exercise 19.5.6. $V(t)$ measures the amount of water (in liters) in a water tank at time t (in hours from midnight last night).

(a) What units does $V'(t)$ have?

(b) Suppose $V(t) = 100 - 2^t$. On average, how much water is in the water tank between $t = 2$ and $t = 3$?

Exercise 19.5.7. $F(t)$ measures the amount of water (in liters per hour) entering or leaving a water tank at time t (in hours from midnight last night). If $F(t)$ is positive, it means the amount of water in the water tank is increasing, while if $F(t)$ is negative, the amount of water is decreasing.

(a) What units does $F'(t)$ have?

From hereon, suppose $F(t) = -10 + t^2$.

(b) On average, how quickly is the volume of water in the tank changing between $t = 1$ and $t = 3$? Make sure to specify if, on average, the amount of water in the tank is decreasing or increasing.

(c) Compute $\int_0^4 F(t)\,dt$.

(d) Give a physical interpretation to the answer from your previous question.

(e) Are you able to tell me how much water is in the water tank at $t = 4$?

Exercise 19.5.8. A gig worker is paid at a rate of $r(t) = 10 + \sin(\frac{1}{12\pi}t)$, where $r(t)$ is in dollars per hour, and t is in hours from midnight.

(a) How much does the worker make if they work from 9 AM to 5 PM?

(b) How does their 9 AM - 5 PM earnings compare to the 9 AM - 5 PM earnings of someone working at a flat rate of 10 dollars an hour?

(Warning: The function $r(t)$ used here was arbitrary, and does not in any way purport to realistically model the wagers of particular gig workers.)

Exercise 19.5.9. Planet X exerts a force of $\frac{100}{x^2}$ Newtons on a box x kilometers away from the center of planet X.

How much work does it take to move the box from 1,000 kilometers away to 10,000 kilometers away from Planet X?

Exercise 19.5.10. Suppose $F(t) = 7t^2 - 9$. Compute

$$\int_3^4 F'(t)\,dt.$$

Exercise 19.5.11. The nucleus of the hydrogen atom exerts a force of

$$\frac{(9 \times 10^9) \times (1.6 \times 10^{-19})^2}{r^2}$$

(in Newtons) on an electron r meters away from the nucleus.

1. It is common to pretend that a typical electron is 5.3×10^{-11} meters away from the nucleus. Pretending this, what is the force exerted by the nucleus on this electron? You do not need to simplify your answer.

2. How much work does it take to move an electron from 5.3×10^{-11} meters away, to 100 meters away from the nucleus?

   Set up the integral. You do not need to compute it.

19.6 Riemann sums and summation notation

Exercise 19.6.1. Write out all the terms in the following summations. For example,

$$\sum_{b=2}^{5} 3b = 3 \times 2 + 3 \times 3 + 3 \times 4 + 3 \times 5.$$

For this problem, you do not need to compute what the sum is.

(a)

$$\sum_{i=3}^{5} \cos(i\pi)$$

(b)

$$\sum_{a=1}^{4} a^2$$

(c)

$$\sum_{n=2}^{6} \frac{1+n}{n^2}$$

(d)

$$\sum_{j=0}^{3} (-1)^j$$

Exercise 19.6.2. (a) Write down, using Σ notation, a Riemann sum approximating the integral

$$\int_1^5 7x\,dx$$

using $n = 4$ rectangles and using either the lefthand or righthand rule (your choice).

(b) Write out the entire summation—that is, convert your Σ notation into a sum of four terms. Note that there *should not* be any x or x_i in your answer. However, you do *not* need to add up the four terms to come up with a single number.

Exercise 19.6.3. (a) Write down, using Σ notation, a Riemann sum approximating the integral

$$\int_1^4 \sin(x)\,dx$$

using $n = 6$ rectangles and using either the lefthand or righthand rule (your choice).

(b) Write out the entire summation—that is, convert your Σ notation into a sum of four terms. Note that there *should not* be any x or x_i in your answer. However, you do *not* need to add up the four terms to come up with a single number.

19.7 Some challenges

Exercise 19.7.1. How do Riemann sums for the function $f(x) = \frac{1}{x}$ from $a = 1$ to $b = t$ help you compute $\ln(t)$?

Exercise 19.7.2. Using the mean value theorem and the fundamental theorem of calculus, show that for any interval $[a, b]$ and a function f, there is some number c between a and b so that $f(c)$ is equal to the average value of f on the interval $[a, b]$.

Exercise 19.7.3. Using Riemann sums, can you explain to me why the statements

(i) You can make $\ln(x)$ as big as you want so long as you make x bigger, and

(ii) You can make the sum $\sum_{n=1}^{L} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{L}$ as big as you want so long as you make L bigger

are either both true, or both false?

Exercise 19.7.4. Using Σ summation notation, write out what the degree 5 Taylor polynomial for a function f is, centered at a point a. As a hint, you can write $f^{(n)}(a)$ to mean the nth derivative of f, evaluated at $x = a$.