Lecture 18

Areas between curves, and average values

We've so far seen how to find areas of regions between (graphs of) functions and the x-axis. But we can make more interesting shapes by looking at regions between graphs of *two* functions.
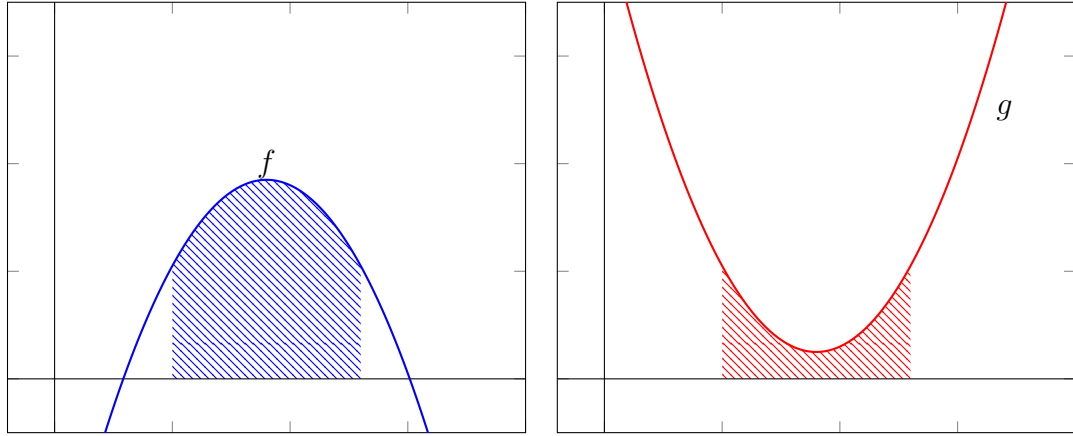
18.1   Areas between two graphs

Below is a picture of two functions, f and g. f is in blue, and is concave down. g is in red, and is concave up.

How would we find the area of the shaded region?
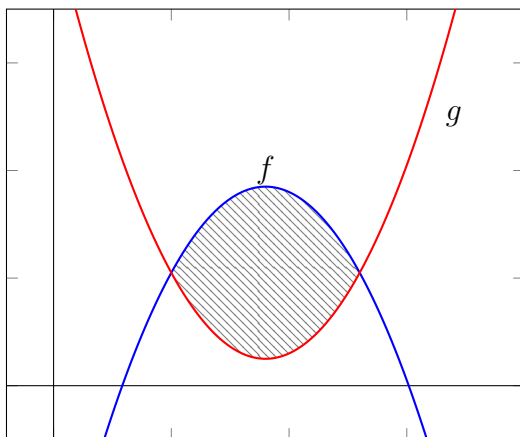
Well, let's look at this region as obtained by taking a big region, and subtracting off another. Observe:

In blue is the area between the graph of f and the x-axis, while in red is the area between the graph of g and the x-axis. Overlaying the pictures, we see that our original region is obtained by removing the red region from the blue region.
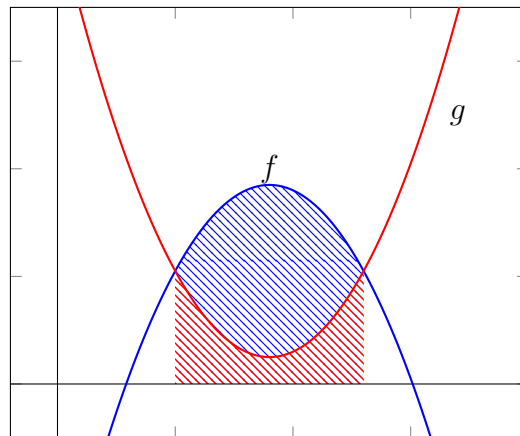
So in this particular example, we can conclude that

$$\text{Area of region between } f \text{ and } g = \int_a^b f \, dx - \int_a^b g \, dx.$$

Here, a and b are where the graphs of f and g intersect; they are the rightmost and leftmost points of the region we seek:
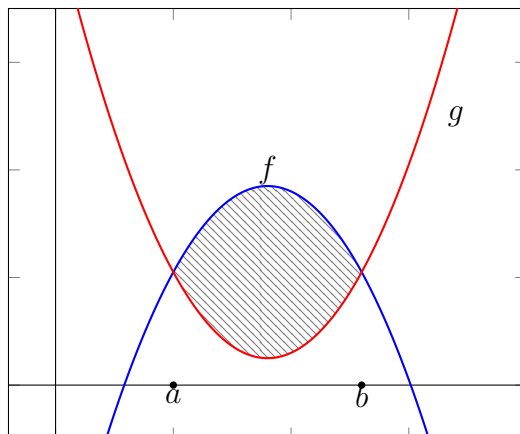
The upshot is as follows:

Proposition 18.1.1. Suppose that the graphs of $f(x)$ and $g(x)$ intersect at the points a and b, with $a < b$. Suppose also that $f(x) \geq g(x)$ for all points x along the interval $[a, b]$. Then the area of the region formed by $f(x)$ and $g(x)$ is given by

$$\int_a^b f(x) - g(x)\, dx.$$

Remark 18.1.2 (Finding a and b). For a problem like this, you may typically be given values of a and b, or you may have to find the values of a and b yourself. To find a and b, you have to solve for the numbers a and b at which $f(a) = g(a)$ and $f(b) = g(b)$.

Example 18.1.3. Find the area between the graphs of the functions $\cos(x)$ and $x^2 - 1 + \cos(x)$.

We must identify where the two functions intersect. This happens when

$$\cos(x) = x^2 - 1 + \cos(x).$$

Solving this equation, we arrive at the conclusion that x must equal -1 or 1. Since $-1 < 1$, we set $a = -1$ and $b = 1$.

Next we must decide which function is larger than (i.e., on top of) the other. We can test this at any point between -1 and 1, so let's try $x = 0$. Then $(0^2) - 1 + \cos(0)$ is less than $\cos(0)$, so we let $\cos(0)$ be the "on top" function. The proposition above tells us to subtract the bottom function from the top function, and integrate form a to b:

$$\int_{-1}^1 (\cos(x)) - \left(x^2 - 1 + \cos(x)\right)\, dx.$$

We can simplify the integrand before integrating:

$$= \int_{-1}^{1} -x^2 + 1 \, dx.$$

Now we conclude
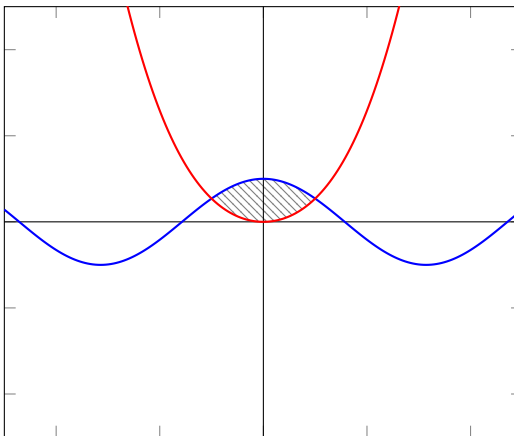
$$\int_{-1}^{1} -x^2 + 1 \, dx = \frac{-1}{3}x^3 + x \Big|_{-1}^{1} \tag{18.1.1}$$

$$= \left(\frac{-1}{3}(1)^3 + (1) \right) - \left(\frac{-1}{3}(-1)^3 + (-1) \right) \tag{18.1.2}$$

$$= \left(\frac{2}{3} \right) - \left(\frac{-4}{3} \right) \tag{18.1.3}$$

$$= 2. \tag{18.1.4}$$

In case you are curious, here is what the region looks like:

18.2   Practice problems for finding areas between curves

Exercise 18.2.1. Determine the area of the region bounded by $y = \frac{8}{x}$, $y = 3x$ and $x = 5$.

Exercise 18.2.2. Determine the area of the region bounded by $x = 2+y^2$, $x = 1-y^2$, $y = 2$ and $y = -3$. (You may want to ask Hiro about this one.)

Exercise 18.2.3. Find the area of the region bounded by $y = x^2 - x - 6$ and $y = 2x + 4$.

Exercise 18.2.4. Find the area of the region bounded by $y = x + 1$ and $y = 9 - x^2$.

Exercise 18.2.5. Find the area of the region bounded by $y = x$ and $y = x^2$.

Exercise 18.2.6. Find the area of the region bounded by $x = 1 - y^2$ and $y = y^2 - 1$.

18.3   Average values

Definition 18.3.1. Let f be a function and choose two real numbers a and b such that $a < b$. Then *the average value* of f over the interval $[a, b]$ is defined to be

$$\frac{\int_a^b f(x)\, dx}{b - a}.$$

Of course, the above is equal to

$$\frac{1}{b - a} \int_a^b f(x)\, dx.$$

Example 18.3.2. A bird travels. The velocity at time t is given by the function $v(t) = t^2$ kilometers per second. What is the average velocity with which the bird was traveling between $t = 0$ and $t = 3$?

Solution. We must compute

$$\frac{1}{b - a} \int_a^b v(t)\, dt = \frac{1}{3 - 0} \int_0^3 t^2\, dt \tag{18.3.1}$$

$$= \frac{1}{3} \left(\frac{1}{3} t^3 \Big|_0^3 \right) \tag{18.3.2}$$

$$= \frac{1}{3} \left(\frac{1}{3} \left((3)^3 - (0)^3 \right) \right) \tag{18.3.3}$$

$$= 3. \tag{18.3.4}$$

The average speed of the bird is 3 kilometers per second.

Remark 18.3.3. The above example should give you an idea of why this really is an average value. Remember that when v is velocity, $\int_a^b v\, dt$ computes the distance traveled. So to divide this value by $b - a$ is to divide total distance traveled by the time elapsed. As you know, this measures how quickly, on average, one is traveling over that time. In other words, this does indeed compute average velocity.

18.4   Average values practice problems

Exercise 18.4.1. Find the average value of $f(x) = 3 + 2x^2$ on the interval $[0, \sqrt{3}]$.

Exercise 18.4.2. An object attached to a (horizontally aligned) spring moves with velocity $v(t) = \sin(t)$.

(a) What is the average velocity of this object over the interval $[0, 2\pi]$?

(b) What is the average velocity of this object over the interval $[0, \pi]$?

Exercise 18.4.3. The number of new infections per day at the beginning of an outbreak can be modeled by the function $f(t) = e^t$, where t is in days and $f(t)$ is in units of (new) infections per day.

(a) At day 10, how many new infections are arising per day? (You can use a calcu-lator if you want a decimal answer.)

(b) Between day 0 and day 10, on average, how many new infections have there been per day? (You can use a calculator if you want a decimal answer.)