# Lecture 17

# $u$ substitution

We saw last time that to find areas (i.e., to compute integrals) we must find antiderivatives.

$u$ substitution is a trick for finding antiderivatives.

## 17.1 $u$ substitution

You can think of $u$ substitution as like a "reverse chain rule." Let me say what I mean.

Suppose $F(x) = g(h(x))$. That is, $F = g \circ h$, so that $F$ is a composition the functions $g$ and $h$. Then you know that

$$F'(x) = g'(h(x)) \cdot h'(x). \tag{17.1.1}$$

That's the chain rule.

In the last lecture, we saw the importance of being able to *work backwards* to find antiderivatives—that is, can you recognize when you see something like $g'(h(x)) \cdot h'(x)$? If so, all you need to do to find the antiderivative is

- Recognize $h$, and
- Take the antiderivative of $g'$. Then to conclude, just
- Set $F = g \circ h$.

**Exercise 17.1.1.** Find an antiderivative for the following functions:

(a) $f(x) = 2x\cos(x^2)$.

(b) $f(x) = \frac{2x}{x^2+3}$

(c) $f(x) = -\sin(\sin(x)) \cdot \cos(x)$.

I am not exactly sure of why—perhaps because it is hard to recognize two derivatives ($g'$ and $h'$) at once—calculus textbooks teach us a technique called *u substitution* to find antiderivatives in situations like this. It can sometimes be confusing, and though I am not a huge fan of $u$ substitution, I will teach it to you in case you find it easier than eye-balling the chain rule.

The way $u$ substitution works is by identifying the $h$ in the equation (17.1.1). For example, consider the indefinite integral

$$\int \cos(x)\sqrt{\sin(x)}dx. \tag{17.1.2}$$

You might recognize a "function within a function," i.e., a composition, in $\sqrt{\sin(x)}$. You might recognize that the "inside function"—$\sin(x)$—has a derivative given by the factor outside the $\sqrt{--}$ symbol, namely the $\cos(x)$ factor. Thus you can verify that the inside function $h(x) = \sin(x)$ says that our integrand is of the form $g'(h(x)) \cdot h'(x)$. In this case, then, we see that $g'$ must be the square root function.

But, rather than thinking this all through, $u$ substitution encourages you to stop thinking and try to do algebra instead. (I am not a fan.) Here is how you do it:

Step One (identifying and substituting $u$): One substitutes the inside function by a variable $u$. You should think of $u$ as a function of $x$. So, for example, a naive re-writing of (17.1.2) gives

$$\int \cos(x)\sqrt{u}dx. \tag{17.1.3}$$

Things look worse right now—there is a $u$ and an $x$ and who knows what in the world this means. Here is the (useful?) confusing part:

**Notation 17.1.2** ($du$)**.** Because $u$ is a function of $x$, we can introduce a new symbol called

$$du$$

that is *defined* to satisfy the following property:

$$du = \frac{du}{dx}dx. \tag{17.1.4}$$

Indeed, note that *if you are allowed to cancel symbols like $dx$* the lefthand side of (17.1.4) can be obtained from the righthand side by "cancelling" the $dx$. I warn you that $du$ and $dx$ are just symbols—they are *not* numbers—so the fraction notation is more misleading than it is useful. You can't just cancel symbols willy-nilly without knowing what they mean. Regardless, $du$—as a symbol—is defined precisely in a way that encourages such dangerous (and, in this case, correct) cancellation.

Or, rearranging (17.1.4), we find

$$dx = \frac{1}{\frac{du}{dx}}\ du. \tag{17.1.5}$$

End of notation.

Step Two (replacing $dx$ by $du$ terms): We plug in $u(x) = \sin(x)$, so that $\frac{du}{dx} = \cos(x)$. Then we can continue to simplify (17.1.3):

$$\int \cos(x)\sqrt{u}dx = \int \cos(x)\sqrt{u}\frac{1}{\frac{du}{dx}}\ du \tag{17.1.6}$$

$$= \int \cos(x)\sqrt{u}\frac{1}{\cos(x)}\ du \tag{17.1.7}$$

$$= \int \sqrt{u}\ du. \tag{17.1.8}$$

Notice that we have used the definition of $du$ to get rid of the $dx$.

Step Three: Take the integral in terms of $u$. What the indefinite integral in (17.1.8) is asking is: Can you find the antiderivative of the square root function? Yes, you can! Moreover, the integral is no longer viewing the integrand as a function of $x$; the "$du$" symbol is telling you to think of the integrand as a function of $u$. Well,

$$\frac{d}{du}(u^{3/2}) = \frac{3}{2}u^{1/2},$$

so we find that

$$\frac{d}{du}\frac{2}{3}(u^{3/2}) = u^{1/2}.$$

In other words, we can solve the indefinite integral in (17.1.8) to find

$$\int \sqrt{u}\ du = \frac{2}{3}u^{3/2}. \tag{17.1.9}$$

And now let's plug back in what $u$ equals; we defined $u$ to be $u(x) = \sin(x)$, so the righthand side of (17.1.9) becomes

$$\frac{2}{3}u^{3/2} = \frac{2}{3}(\sin(x))^{3/2} = \frac{2}{3}\sqrt{\sin(x)^3}.$$

Indeed, you can check that this function of $x$ is an antiderivative of our original function $\cos(x)\sqrt{\sin(x)}$.

Here is the **summary of $u$ substitution**:

$$\int g'(h(x))h'(x)dx = \int g'(u)du.$$

In the end, if you find the integral $\int g'(u)du = g(u)$, make sure you substitute back in $h(x) = u(x)$ to get

$$\int g'(h(x))h'(x)dx = g(h(x)).$$

**Exercise 17.1.3.** Compute the following indefinite integrals:

(a) $\int \frac{1}{x}\sqrt{\ln(x)}\,dx$

(b) $\int 3x^2\cos(x^3)\,dx$

(c) $\int x^2\cos(x^3)\,dx$

(d) $\int \sin(x)\cos(\cos(x))\,dx$

(e) $\int x^3 e^{x^4}\,dx$

## 17.2 Application: The integral of $\tan(x)$

Here is (what I think is) a good application of $u$ substitution.

**Exercise 17.2.1.** Find

$$\int \tan(x)dx.$$

Let's note

$$\int \tan(x)dx = \int \frac{\sin x}{\cos x}dx = \int \sin(x)\cdot\frac{1}{\cos(x)}dx.$$

We note that $\sin(x)$ is (almost) the derivative of $\cos(x)$—it's off by a sign. But it almost looks like we can take

$$g(x) = \frac{1}{x}, \qquad h(x) = \cos(x),$$

for then

$$g(h(x))h'(x) = \frac{1}{\cos(x)}\cdot(-\sin(x)).$$

So we have that

$$\int \sin(x) \cdot \frac{1}{\cos(x)} dx = -\int (-\sin(x)) \cdot \frac{1}{\cos(x)} dx. \tag{17.2.1}$$

Letting $u = \cos(x)$, we have that

$$du = -\sin(x)dx, \qquad dx = \frac{du}{-\sin(x)}.$$

Hence (17.2.1) becomes

$$-\int (-\sin(x)) \cdot \frac{1}{\cos(x)} dx = -\int (-\sin(x)) \cdot \frac{1}{u} \cdot \frac{du}{-\sin(x)} \tag{17.2.2}$$

$$= -\int \frac{1}{u} du \tag{17.2.3}$$

But you know how to integrate $\frac{1}{u}$; the antiderivative is $\ln(|u|)$. Hence we have

$$-\int \frac{1}{u} du = -\ln(|u|) + C.$$

Now, let's remember that $u(x) = \cos(x)$, so plugging this in, we have

$$\int \tan(x) dx = -\int \frac{1}{u} du = -\ln(|u|) + C = -\ln(|\cos(x)|) + C. \tag{17.2.4}$$

Here is one more simplification we can make: Remember the formula

$$a \ln(b) = \ln(b^a).$$

(If you don't remember it, you should verify it using what you know about exponent laws and the definition of ln!) In particular,

$$-\ln(b) = \ln(\frac{1}{b}).$$

Thus, we can further modify (17.2.4) to become

$$\int \tan(x) dx = \ln(\left|\frac{1}{\cos(x)}\right|) + C.$$

Or, if you like secant, which is defined by $\sec(x) = 1/\cos(x)$, you can rewrite this as

$$\int \tan(x) dx = \ln(\left|\sec(x)\right|) + C.$$

**Remark 17.2.2.** If you prefer the "eyeball" method, you could have recognized that $\tan(x)$ is of the form $\sin(x) \times$ something, and that this something has $\cos(x)$ in it. Thus you could be inspired to use the (reverse) chain rule.

$$\sin(x) \cdot \frac{1}{\cos(x)} = h'(x) \cdot g'(h(x)).$$

You recognize now that $g'(x)$ has to be $\frac{1}{x}$, so that $g(x)$ has to be $\ln|x|$. Then, by the (reverse) chain rule,

$$\int g'(h(x))h'(x)dx = g(h(x)) + C = \ln|\frac{1}{\cos x}| + C.$$

I much prefer this method, but there are uses for $u$ substition in one's life, so if you prefer to solve problems using $u$ substitution (which will require you to get used to manipulating equations like $du = \frac{du}{dx}\, dx$), go for it!

## 17.3 Using $u$ substitution to compute integrals

$u$ substitution isn't just for computing antiderivatives; it also allows you to compute integrals!

**Fact.** If $u(x) = h(x)$, then

$$\int_a^b g'(h(x))h(x)dx = \int_{u(a)}^{u(b)} g'(u)du. \tag{17.3.1}$$

**Example 17.3.1.** Let's evaluate

$$\int_1^4 \frac{2x}{1+x^2}dx.$$

If I want to use $u$ substitution, I recognize that $2x$ is the derivative of $1 + x^2$. So I will set $u(x) = 1 + x^2$, so that $du = 2xdx$. Then

$$\int \frac{2x}{1+x^2}\, dx = \int \frac{2x}{u} \cdot \frac{1}{2x}\, du = \int \frac{1}{u} du.$$

What the fact (17.3.1) tells us is that we can evaluate the definite integral in using the $u$ variable form of the integral:

$$\int_1^4 \frac{2x}{1+x^2}dx = \int_{u(1)}^{u(4)} \frac{1}{u}du.$$

So we find

$$\int_{u(1)}^{u(4)} \frac{1}{u} du = \ln|u| \Big|_{u(1)}^{u(4)} \tag{17.3.2}$$
$$= \ln|u| \Big|_{1+1^2}^{1+4^2} \tag{17.3.3}$$
$$= \ln|u| \Big|_{2}^{17} \tag{17.3.4}$$
$$= \ln|17| - \ln|2| \tag{17.3.5}$$
$$= \ln\frac{|17|}{|2|} \tag{17.3.6}$$
$$= \ln\frac{17}{2}. \tag{17.3.7}$$

If we want, we could have computed this without using $u$ substitution. Again recognizing that if $h(x) = 1 + x^2$, then $h'(x)$, we have that the integrand is equal to $h'(x) \cdot \frac{1}{h(x)}$. Thus we want $g'(x) = \frac{1}{x}$, which has integral $g(x) = \ln|x|$. We conclude

$$\int_1^4 g'(h(x))h'(x)\,dx = g(h(x)) \Big|_1^4 \tag{17.3.8}$$
$$= \ln|1+4^2| - \ln|1+1^2| \tag{17.3.9}$$
$$= \ln|17| - \ln|2| \tag{17.3.10}$$
$$= \ln\frac{17}{2}. \tag{17.3.11}$$

**Exercise 17.3.2.** Compute the following.

(a)

$$\int_0^1 x(x^2-1)^5 dx.$$

(b)

$$\int_0^{1/12} \frac{1}{\sqrt[3]{1-6x}} dx.$$

(c)

$$\int_2^3 xe^{x^2}\ dx$$

(d)

$$\int_0^1 x(x^2-1)^5 dx.$$

(e)

$$\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{\sin^2(x)}\, dx.$$