Lab exercises involving u substitution

Exercise 17.3.3. Find an antiderivative to each of the following functions.

(a) $x^4(x^5 - 9)^{3/2}$

(b) $\frac{2x}{x^2-9}$

(c) $\frac{\cos(x)}{\sin(x)}$

(d) $\frac{1}{x \ln(x)}$ (Hint: Try $u = \ln x$.)

(e) $x^{1/2} \cos(x^{3/2})$

(f) $\frac{2x-9}{x^2-9x}$

(g) $\cos(x)e^{\sin(x)}$

(h) xe^{3x^2}

(i) $x(3x^2 - 9)^2$

(j) $\frac{\sin(\ln x)}{x}$

Possible solutions. (a) $\frac{2}{25}(x^5 - 9)^{5/2}$

(b) $\ln(|x^2 - 9|)$

(c) $\ln|\sin(x)|$

(d) $\ln(|\ln x|)$

(e) $-\frac{2}{3}\sin(x^{3/2})$

(f) $\ln(|x^2 - 9x|)$

(g) $e^{\sin(x)}$

(h) $\frac{1}{6}e^{3x^2}$

(i) $\frac{1}{18}(3x^2 - 9)^3$

(j) $-\cos(\ln(x))$

Remember that u substitution works well for functions that look like (i) A product of two functions, one of which is a composition, and (ii) the other factor looks like the derivative of the inside-function of the composition. Other times, u substitution works well if you just try to "substitute a complicated term with u" and see what happens.

As an example, for the first problem, we see a composition $(x^5 - 9)^{3/2}$. This is a composite of $x^5 - 9$ and $x^{3/2}$, so we try setting $u = x^5 - 9$ for the inside function.

We are left to compute the indefinite integral

$$\int x^4(x^5-9)^{3/2}\,dx = \int x^4 u^{3/2}\,dx = \int \frac{1}{5}u^{3/2}\,du$$

where the last equality used the equality $du = \frac{du}{dx}dx = 5x^4 dx$. We know how to integrate $u^{3/2}$ using (the reverse of) the power rule:

$$\int \frac{1}{5}u^{3/2}\,du = \frac{1}{5}\cdot\frac{2}{5}u^{5/2} + C = \frac{2}{25}u^{5/2} + C = \frac{2}{25}(x^5-9)^{5/2} + C.$$

So an antiderivative is given by $\frac{2}{25}(x^5-9)^{5/2}$. □

Exercise 17.3.4 (These are a bit harder.)**.** Find an antiderivative for the following functions

(a) $\frac{1}{3\sqrt{x}+3x}$ (Hint: Take $u = 1+\sqrt{x}$.)

(b) $\frac{x}{9x-1}$ (Hint: Take $u = 9x-1$. Then do some algebra.)

(c) $\frac{x-1}{x+2}$

(d) $\frac{x+3}{(x-1)^{3/2}}$. (Hint: Take $u = x-1$.)

(e) $\frac{x+1}{(x-2)^{5/2}}$.

(f) $\frac{1}{(x-2)^{5/2}}$.

(g) $x^2\sqrt{1-x}$. (Hint: Take $u = 1-x$. Then do some algebra.)

(h) $x^3\sqrt{x^2-2}$.

(i) $x^7(x^4-1)^{1/4}$.

(j) $x(x-3)^{1/3}$

(k) $4x(7x-1)^{1/5}$.

Possible solutions. (a) Note $u = 1+\sqrt{x}$ gives $du = \frac{1}{2}x^{-1/2}dx$ so that $dx = 2x^{1/2}du$.

$$\int \frac{1}{3\sqrt{x}+3x}\,dx = \int \frac{1}{3\sqrt{x}(1+\sqrt{x})}\,dx = \int \frac{1}{3\sqrt{x}\cdot u}\,dx = \int \frac{1}{3\sqrt{x}\cdot u}2x^{1/2}\,du$$
$$= \int \frac{2}{3u}\,du = \frac{2}{3}\ln|u| + C = \frac{2}{3}\ln(|1+\sqrt{x}|) + C.$$

Instead, one could also note that the "complicated" term is $\sqrt{x}$, so we could try $u = \sqrt{x}$ out of a desire to get rid of the wonky term. Then $du = \frac{1}{2}x^{-1/2}\,dx$ and

$$\int \frac{1}{3\sqrt{x}+3x}\,dx = \int \frac{1}{3u+3u^2}\,dx = \int \frac{1}{3u+3u^2}\,dx = \int \frac{1}{3u+3u^2}2u\,du$$
$$= \frac{2}{3}\int \frac{u}{u+u^2}\,du \; = \frac{2}{3}\int \frac{1}{1+u}\,du \; = \frac{2}{3}\ln(|1+u|) + C = \frac{2}{3}\ln(|1+\sqrt{x}|) = C.$$

By the way, we have the absolute value signs in there to be safe, but $\sqrt{x}$ is always positive so $1+\sqrt{x} = |1+\sqrt{x}|$.

(b) Setting $u = 9x - 1$ so $du = 9dx$ and $x = \frac{1}{9}(u+1)$ we have

$$\int \frac{x}{9x-1}\,dx = \int \frac{\frac{1}{9}(u+1)}{u}\frac{1}{9}\,du = \frac{1}{81}\int \frac{u+1}{u}\,du = \frac{1}{81}\int \frac{u}{u} + \frac{1}{u}\,du$$

$$= \frac{1}{81}\int 1+\frac{1}{u}\,du = \frac{1}{81}(u+\ln|u|)+C = \frac{1}{81}(9x-1+\ln|9x-1|)+C = \frac{1}{9}x+\frac{\ln|9x-1|}{81}+C$$

(c) Life would be easier if the denominator were simpler, so let's just try $u = x+2$. Then $du = dx$ and $x - 1 = u - 3$. We find

$$\int \frac{x-1}{x+2}\,dx = \int \frac{u-3}{u}\,du = \int \frac{u}{u} - \frac{3}{u}\,du$$

$$= \int 1 - \frac{3}{u}\,du = u - 3\ln|u| + C = x + 2 - 3\ln|x+2| + C = x - 3\ln|x+2| + C$$

□

Exercise 17.3.5 (Challenge)**.** Suppose you know f is a function for which $\int_1^4 f(x)\,dx = 8$. What can you say about

$$\int_1^2 xf(x^2)\,dx?$$

Possible solution. Letting $u = x^2$, we see that

$$\int_1^2 xf(x^2)\,dx = \int_1^4 \frac{1}{2}f(u)\,du = \frac{1}{2}\int_1^4 f(u)\,du = \frac{1}{2}\times 8 = 4.$$

□

Exercise 17.3.6. It is estimated that the average person expels about 500 liters of carbon dioxide per day, or 182,500 liters per year. So, the rate of carbon dioxide produced by respiration is about

182,500 liters per person per year.

We can model the world's human population using the function

$$P(t) = 7.3 \times 2^{\frac{t}{50}}$$

where t is measured in years, P is measured in *billions* of people, and we'll take January 1, 2015 as our starting point $t = 0$.

According to this model, how many liters of carbon dioxide will have been emitted by human beings (by breathing alone) between January 1, 2015 and January 1, 2025?

You should express an exact form of your answer (it will involve natural logs).

Possible solution. The rate at which carbon dioxide is being released at time t is measured by

$$182500 \times P(t) = 182500 \times 7.3 \times 2^{t/50}$$

in units of liters per year. To find out how many liters are emitted total from $t = 0$ to $t = 10$, we integrate:

$$\int_0^{10} 182500 \times 7.3 \times 2^{t/50}\, dt.$$

Now note that

$$2^{t/50} = (e^{\ln 2})^{t/50} = e^{\frac{\ln 2}{50} t}.$$

And the indefinite integral is

$$\int e^{\frac{\ln 2}{50} t}\, dt = \frac{50}{\ln 2} e^{\frac{\ln 2}{50} t} + C. = \frac{50}{\ln 2} 2^{\frac{t}{50}} + C.$$

Having found the antiderivative, we can now compute:

$$\int_0^{10} 182500 \times 7.3 \times 2^{t/50}\, dt = 182500 \times 7.3 \times \int_0^{10} 2^{t/50}\, dt = 182500 \times 7.3 \times \left. \frac{50}{\ln 2} 2^{\frac{t}{50}} \right|_0^{10}$$

$$= 182500 \times 7.3 \times \frac{50}{\ln 2}\left(2^{10/50} - 2^{0/50}\right) = 182500 \times 7.3 \times \frac{50}{\ln 2}\left(2^{1/5} - 1\right).$$

□

Exercise 17.3.7. (The following model is entirely fictional, though Mechanical Turk is a real thing. I do not know what the average hourly wage on Mechanical Turk is.)

On the website Mechanical Turk, we can model the number of people $P(t)$ working at a particular hour t using the function

$$P(t) = 12000 + 10000 \sin(\frac{2\pi}{24} t)$$

where t is measured in hours from midnight.

On average, a worker for Mechanical Turk is paid 1.3 dollars per hour. (So the website pays 1.3 dollars per hour per worker.)

Based on this, how much money did the website Mechanical Turk have to pay its workers between 9 AM and 6 PM?

You should express an exact form of your answer (it will involve sines or cosines) before using a calculator to give a decimal form.

Exercise 17.3.8. Some rideshare apps use "surge" pricing to change the cost of a hailed ride. Let's assume that Company U pays its drivers a wage that depends on this surge pricing, and that we can model worker wages by the function

$$W(t) = 9 + 3\cos(\frac{2\pi}{12}(t-8)).$$

Here, W is measured in dollars per hour and t is measured in hours past midnight. (So for example, at 8 AM, a driver is earning 12 dollars per hour. At 2 PM, a driver is earning 6 dollars per hour.)

Based on this model, if a driver works from 9 AM to 5 PM, how much money do they earn from Company U?

(A similar model can be used to calculate costs of electricity, which typically fluctuate based on time of day, though with electricity, we must also couple the "surge" pricing with the amount of electricity we actually use to determine price.)