Lecture 16

Integration and the Fundamental Theorem of Calculus

Given a function f, along with an interval $[a, b]$, we saw last time that we can *approximate* the area under the graph of f.[1] We did this by choosing n rectangles, so that their widths were given by $(b - a)/n$, and by choosing a height of each rectangle as dictated by f. We ended up with a summation that looked like

$$\sum_{i=0}^{n-1} f(x_i) \frac{b-a}{n} \qquad \text{(Lefthand rule).}$$

Now, these approximations should get better the more rectangles that we use—that is, the bigger n is. In this class, we will define the integral as follows:

Definition 16.0.1 (The integral)**.** The integral of f from a to b is defined to be:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \frac{b-a}{n}.$$

We denote this limit by

$$\int_a^b f(x)dx.$$

In words: This is the limit of the numbers we obtain from the Riemann sum when we let n grow larger and larger.

[1] "Under" is informal; remember, this is the area between the graph of f and the x-axis, computed with "sign," where regions under the x-axis are declared to have *negative* areas.

Remark 16.0.2. You might wonder if we could define an integral to be computed not as the limit of *equal interval* subdivisions of $[a, b]$, but using arbitrary subdivisions and taking the limit as these arbitrary subdivisions get finer and finer. Indeed, you can; but setting up this theory would distract us from the main thread of this course. A proper course in analysis may deal with a proper way to set up the so-called "Riemann integral" (though some analysis classes skip over this in favor of a Lebesgue integral).

16.1 Getting used to, and reading, the notation

This is the first time you've seen the notation

$$\int_a^b f(x)dx.$$

Let's dissect this.

First, the above collection of symbols can be described as:

"The integral of $f(x)$ from a to b."

If you were reading the symbols out loud like an automated reader, you would say "the integral from a to b of f of x, dx." [2]

Second: $\int$ is called "the integral symbol." And a and b are called the *bounds* of the integral. The interval $[a, b]$ is sometimes called the *region of integration.*

Third: $f(x)$ is referred to as the *integrand* of the integral.

Finally, let's talk about where this notation comes from. Well, we saw a Riemann sum:

$$\sum_{i=1}^{n} f(x_i)(b-a)/n$$

which we could rewrite (by thinking of $(b-a)/n$ as a change in x) as

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

Now imagine being supremely lazy and neglecting to write the super and subscripts for Σ:

$$\sum f(x_i)\Delta x.$$

[2] Analogously, you would describe $f'(x)$ as the derivative of f at x, but if you read the symbols out loud, you would say "f prime of x".

Remember I told you earlier that Σ is a Greek letter that turned into S. Imagine being so lazy that you start writing your S really quickly, until your writing started to look like this:

$$\int f(x_i)\Delta x.$$

At this point you've become so lazy that you don't know that the i mean any more, so let's drop that:

$$\int f(x)\Delta x.$$

And Δ is a Greek letter that has now-a-days turned into D. For no good reason (except to have a lower-case letter connote something smaller than an upper-case letter), let's make it lower-case:

$$\int f(x)dx.$$

You should remember that you're taking the area between $x = a$ and $x = b$, so let's at least remember that:

$$\int_a^b f(x)dx.$$

And this is one way to reason out how this notation came about.

Remark 16.1.1. This notation actually has very good reasons to exist, especially because the symbol dx actually has an incredibly fancy meaning in the math community – it's a differential form, and one can always integrate differential forms. But we're not ready to confront that meaning in this class. (Nor will we be ready until you have had enough multivariable calculus, which is a course you would take two semesters from now if you wished.)

16.2 Definitions versus intuitions

Remember that I make a hubbub about what is a definition, and what is an intuition. Here is a table of intuitions and definitions for our two most important ideas:

Why do I make a hubbub? Intuition is how you should think about things: They guide you in solving problems. *Definitions* give content to your intuition, and allow you to actually *prove* things. For example, even if you have an intuition for derivatives, you would be hard-pressed to prove that $(\sin x)' = \cos x$ without the definitions of both limit and of derivative!

I emphasize this. Remember, the equation $(\sin x)' = \cos x$ isn't true just because some teacher told you it was; it's true because you can *prove* it, and the proof doesn't consist of people waving their hands about what the slope of the tangent line should

Term	Definition	Intuition
Derivative	The limit of a difference quotient[a]	The **slope** of the line tangent to the graph of f at x.
Integral	The limit of values of Riemann sums[b]	The signed **area** between the graph of f and the x-axis.

[a] We define $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
[b] We have defined $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\frac{b-a}{n}$.

be—the proof consists of using definitions and logical deductions to get from Step A to Step B. This is the heart of mathematics.

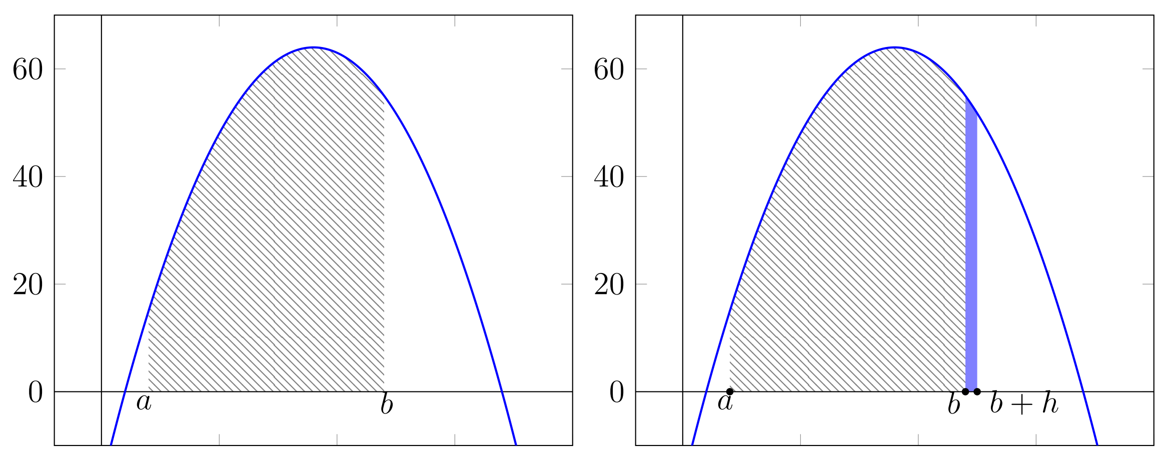
That being said, just as with derivatives, I will only ask you to prove *some* of the rules of integration, and other rules, I will (hypocritically) ask you to just take them on faith. And, as I mentioned before, your *intuition* of thinking about integral as area will help you most.

16.3 The rate of change of area

Let's say you have a function $f(x)$, and you want to compute

$$\int_a^b f(x)dx.$$

As I've advocated, let's think of this as the area. Now let's say you want to nudge the region of integration just a little bit bigger—say by some number h.



In other words, suppose we want to compare

$$\int_a^b f(x)dx \qquad \text{with} \qquad \int_a^{b+h} f(x)dx.$$

The difference is (using our intuition) the area of $f(x)$ between b and $b+h$—this is the darker shaded region in the picture above. In other words, the difference is

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx = \int_b^{b+h} f(x)dx = \text{area of darker shaded region.} \quad (16.3.1)$$

Exercise 16.3.1. (a) By approximating the shaded region using a single rectangle of width h and using the lefthand rule for this single rectangle, approximate

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx.$$

(b) Using your answer from part (a), write down an approximation for

$$\frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h}.$$

Intuitively, would you expect your approximation to be better or worse as h shrinks?

(c) Based on part (b), make a guess as to what

$$\lim_{h\to 0} \frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h}$$

should be in term of f and b.

(d) Note that the area $\int_a^b f(x)dx$ should change if we change the bounds of integration—and in particular, if we change b. So, keeping a unchanged for now, let's define a function F as follows:

$$F(b) = \int_a^b f(x)dx.$$

What can you say about the derivative of F at b?

16.4 A guide for the previous exercise

How can we approximate the area of the grey region? As the exercise suggests, let's use a single rectangle, and the lefthand rule. Then the grey region can be approximated by a single rectangle of width h and height $f(b)$. So

$$\int_b^{b+h} f(x)dx \approx \text{height} \times \text{width} = f(b) \cdot h. \tag{16.4.1}$$

Here, the symbol $\approx$ means "approximately equals." Putting (16.3.1) and (16.4.1) together, we find:

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx \approx f(b) \cdot h.$$

So, dividing both sides by h, we have that

$$\frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h} \approx f(b).$$

Of course, if h is a tiny number, this approximation should get better and better. In other words, that $\approx$ symbol will behave more like an $=$ symbol as h shrinks. Hmm. We've learned what it means to let h approach 0—we should take a limit. Thus, knowing that $\approx$ should become closer and closer to becoming an equals sign as h goes to 0, we seem to get

$$\lim_{h \to 0} \frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h} = f(b). \tag{16.4.2}$$

Remark 16.4.1. Let's take a moment to parse this. This is saying the following: We can think of area—i.e., of $\int_a^b f(x)dx$ as something that depends on b.[3] And (16.4.2) is telling us that the *rate of change at* b—that is, the derivative of the area function at b—seems to be very close to being $f(b)$ itself.

Let's really hone in on this observation. Let's say $F(b)$ is the function that tells us the area of f between a and b. Then (16.4.2) seems to be telling us that

$$\frac{dF}{db}(b) = f(b).$$

In other words, whatever function F measures *area*, it seems to be a function whose derivative recovers f.

[3] Intuitively, if we increase b, we are increasing the region of integration, and the area we're interested in clearly changes. So the number, area, changes depending on b.

16.5 The fundamental theorem of Calculus

All the “seems” and “≈” and “approximates” on the previous page was maddeningly imprecise, suggestive, incomplete. Thanks to Isaac Newton and Gottfried Wilhelm Leibniz, it turns out that what “seems” actually “is.”

Theorem 16.5.1 (The fundamental theorem of calculus)**.** Let $F(x)$ be any function such that

$$F'(x) = f(x).$$

Then, for reasonable functions[4] on the interval $[a, b]$, we have:

$$\int_a^b f(x)dx = F(b) - F(a).$$

A function $F(x)$ such that $F'(x) = f(x)$ is called an *antiderivative* of $f(x)$. What the fundamental theorem says is that we can compute an integral of f (which was defined in a way that had nothing to do with derivatives!) by finding antiderivatives of f.

16.6 Exercises

16.6.1 Finding antiderivatives

The fundamental theorem of calculus tells us we can compute areas involving f by finding functions F for which $F' = f$. Such F are called an *antiderivative* of f. So let’s practice this process of finding antiderivatives.

Exercise 16.6.1. (a) Let $f(x) = 3$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(b) Let $f(x) = 5x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(c) Let $f(x) = x^3$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(d) Let $f(x) = \sin x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(e) Let $f(x) = \cos x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

[4] “Reasonable” is not a technical term. In this class, we will only encounter reasonable functions. If f is differentiable, or even just continuous, f is a “reasonable” function. There are many ways to demand that f be reasonable, with a technical assumption being that f is “Riemann integrable.” We won’t need this fanciness in our course.

(f) Let $f(x) = e^x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(g) Let $f(x) = 1/x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(h) Let $f(x) = x^{4/3}$. Find a function $F(x)$ so that $F'(x) = f(x)$.

Solutions to Exercise 16.6.1. (a) We have seen functions whose derivatives are just constants—for example, if $F(x) = 7x$, then we know $F'(x) = 7$. So if we want $F'(x)$ to equal 3, we can take $F(x) = 3x$.

However, we could also choose $F(x) = 3x + 13$. For then the derivative is again $F'(x) = (3x') + (13)' = 3 + 0 = 3$. Indeed, we could add any number (constant) to $3x$ to obtain a function that satisfies $F'(x) = f(x)$.

(b) Let $f(x) = 5x$. Find a function $F(x)$ so that $F'(x) = f(x)$. We know that the derivative of x^2 is $2x$, so if we just multiply x^2 by the correct multiple, we can engineer the derivative to become $5x$. So let's try

$$F(x) = \frac{5}{2}x^2.$$

Then by the power rule, we indeed find that $F'(x) = \frac{5}{2} \times 2 \times x = 5x$.

And, just as in the previous problem, if we add a constant, so we try

$$F(x) = \frac{5}{2}x^2 + 999$$

for example, then we still have $F'(x) = 5x^2$.

Do you see a pattern? There are *infinitely many* possible choices for F, always. And they'll always differ by some constant.[5]

(c) We know that x^4 has derivative $4x^3$. So just as in the previous problem, let's try

$$F(x) = \frac{1}{4}x^4.$$

Then $F'(x) = \frac{1}{4} \cdot 4x^3 = x^3$. And, as before, we can add any constant to $\frac{1}{4}x^4$ to obtain another function whose derivative is given by $f(x) = x^3$. For example,

$$F(x) = \frac{1}{4}x^4 + \pi$$

is a valid solution.

[5]In fact, we proved after the mean value theorem that if two functions have the same derivative, the two functions must differ by a constant.

(d) Let $f(x) = \sin x$. As before, we must utilize our knowledge of derivatives to realize that $F(x) = -\cos(x)$ is a valid choice. And, as before, we can also take something like $F(x) = -\cos(x) + 23$ (or any constant) as a viable candidate.

(e) Let $f(x) = \cos x$. Then we can take $F(x) = \sin x$, plus any constant we desire.

(f) Let $f(x) = e^x$. We can take $F(x) = e^x$, plus any constant we desire.

(g) Let $f(x) = 1/x$. We can take $F(x) = \ln x$, plus any constant we desire.

(h) Take $F(x) = \frac{3}{7}x^{7/3}$ (plus any constant number).

□

Notation 16.6.2 (plus C)**.** At this point, you may be tired of me telling you how we can add any constant we want. For this reason, we will often write

$$F(x) = \ln x + C \tag{16.6.1}$$

instead of writing out "$F(x)$ could be $\ln x$ plus any constant we desire." The capital C stands for "constant."

This is a very sloppy and ambiguous notation. It is similar to the use of $\pm$ int he quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

x does *not* simultaneously equal everything on the right hand side. The quadratic formula is lazy notation for saying x could equal the fraction on the right given by taking $\pm$ to be $+$, or another possible value of x is to take the righthand side with $\pm$ being set to $-$.

Likewise, the $+C$ in (16.6.1) means an antiderivative of f could be given by $\ln x + 3$, or by $\ln x + \pi$, or by $\ln x + e$, or any possible value of C.

16.6.2 Finding integrals

Now let's use the fundamental theorem of calculus to compute integrals (i.e., areas).

Exercise 16.6.3. (a) Let $f(x) = 3x^2$. Find the area of $f(x)$ along the interval $[1, 4]$.

(b) Let $f(x) = e^x$. Find the area of $f(x)$ along the interval $[0, \ln 3]$.

(c) Let $f(x) = \cos x$. Find the area of $f(x)$ along the interval $[0, \pi/4]$.

Solutions to Exercise 16.6.3. (a) Let $f(x) = 3x^2$. Find the area of $f(x)$ along the interval $[1, 4]$.

In this problem, $[a, b] = [1, 4]$—that is, $a = 1$ and $b = 4$. The area is computed by finding the integral

$$\int_a^b f(x)dx = \int_1^4 3x^2 dx.$$

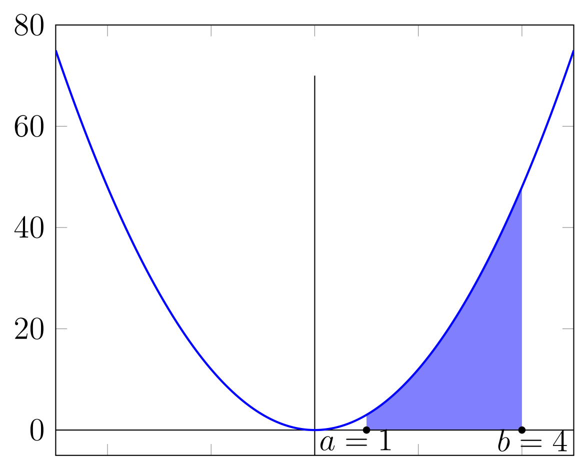
(Note that we never remove the "dx" when we solve problems; we'll see a use for this later on.) By the fundamental theorem of calculus, we know that

$$\int_1^4 3x^2 dx = F(4) - F(1)$$

if F is any function satisfying $F'(x) = 3x^2$. Well, we can find such a function. Let $F(x) = x^3$. So, by the fundamental theorem of calculus (FTC), we have

$$\int_1^4 3x^2 dx = F(4) - F(1) = (4)^3 - (1)^3 = 64 - 1 = 63.$$

How cool is that? You just proved that the region shaded below:



has area given by 63!

As we see from above, the steps to finding the area/integral $\int_a^b f(x)dx$ are: (1) Find an antiderivative for f, then (2) plug in b and a into the antiderivative, and take difference.

(b) Let $f(x) = e^x$. Find the area of $f(x)$ along the interval $[0, \ln 3]$. Again, area is given by

$$\int_0^{\ln 3} f(x)dx.$$

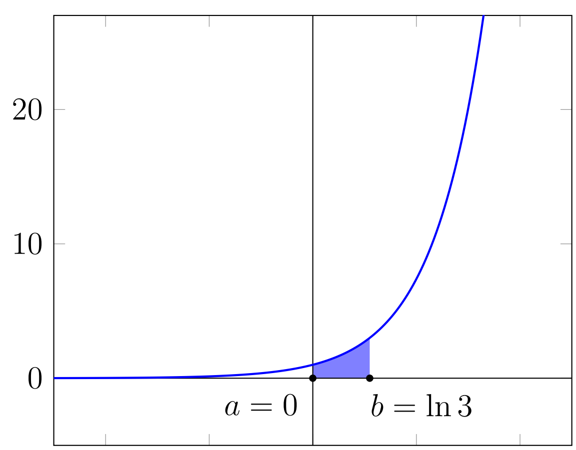
By the fundamental theorem of calculus, if we find some $F(x)$ such that $F'(x) = f(x)$, then

$$\int_0^{\ln 3} f(x)dx = F(\ln 3) - F(0).$$

Well, e^x is its own derivative, so we can take $F(x) = e^x$. Then

$$\int_0^{\ln 3} f(x)dx = F(\ln 3) - F(0) = e^{\ln 3} - e^0 = 3 - 1 = 2.$$

This proves that the region shaded below

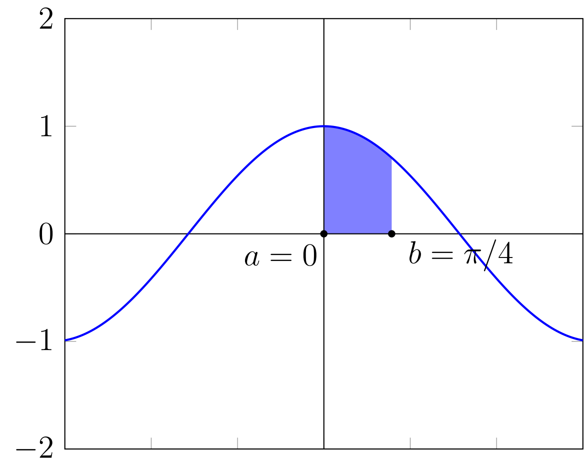


has area given by 2.

(c) Let $f(x) = \cos x$. Find the area of $f(x)$ along the interval $[0, \pi/4]$. Let $F(x) = \sin x$. Then

$$\int_0^{\pi/4} \cos(x)dx = F(\pi/4) - F(0) = \sin(\pi/4) - \sin(0) = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.$$

So you've proven that the area of the region shaded below



is $\frac{\sqrt{2}}{2}$.

□

16.7 A warning about the integral of $1/x$

As you know, the derivative of $\ln(x)$ is $1/x$. So you're tempted to say that $\ln(x)$ is the antiderivative of $1/x$. But a word of caution!

$\ln(x)$ is only defined when x is a positive number. (After all, e^{blah} can only ever equal a positive number.) So it couldn't possibly be the (whole) antiderivative of $1/x$, because $1/x$ also knows how to take in negative numbers.

Well, here is a new function that knows how to take in negative numbers:

$$\ln(|x|).$$

If you like, this function also has a piecewise definition:

$$\ln(|x|) = \begin{cases} \ln(x) & x > 0 \\ \ln(-x) & x < 0. \end{cases}$$

Now, by the chain rule,

$$(\ln(-x))' = -1 \cdot \frac{1}{-x} = \frac{1}{x}.$$

So even for negative values of x, we find that

$$\ln(|x|)' = \frac{1}{x}.$$

So, be warned: **An antiderivative of $\frac{1}{x}$ is** $\ln(|x|)$ (possibly plus some constant).

16.8 Applications of integration

When we studied derivatives, we saw that "the slope of the tangent line at a" had to do with "the rate of change at a." In other words, the geometric idea of slope had real-world implications (by allowing us to understand how things are changing at a given moment).

Likewise, now we'll study how the geometric notion of "area" can actually represent meaningful quantities in the real world.

16.8.1 Recollection on units of derivatives

First, let's talk a little about units (like meters, seconds, liters, dollars et cetera).

Let's say that $f(t)$ is a function telling us the *position* of something—say, in meters—away from some reference point. Let's say that the input variable t is time, measured in seconds. Then when we compute the slope of a secant line

$$\frac{f(t+h)-f(t)}{h}$$

the physical quantity in the numerator is measured in meters, while the quantity in the denominator is measured in seconds. In other words, the above slope has natural units, called

$$\frac{\text{meters}}{\text{seconds}}$$

otherwise known as meters per second. And indeed, these are the units with which we measure speed and velocity. (If f were to output a number in terms of miles, and if t were measured in hours, the slope would have units of miles-per-hour.) This is confirmation, using the physical idea of units, that the derivative of a position-in-time function can be interpreted as velocity.

Example 16.8.1. If $v(t)$ is a function whose input variable t is measured in seconds, and whose output value $v(t)$ is the velocity of something in motion, measured in meters-per-second, then the units for the derviative $v'(t)$ would be in

$$\frac{\text{meters-per-second}}{\text{seconds}}.$$

In other words, the units would be "meters per second per second," often abbrevaited as "meters per second squared," or m/s^2. If you have taken a physics class, you may recognize this as a unit for acceleration—it measures how velocity is changing with respect to time.

Example 16.8.2. If $P(d)$ is a function that tells you the water pressure underwater (measured in millibars) at depth d (measured in meters), then the derivative of P with respect to d would have units

$$\frac{\text{millibars}}{\text{meter}}$$

or millibars-per-meter. This tells you the rate of change with pressure with respect to depth.

16.8.2 The integral of velocity is distance (or position)

Let's say that $v(t)$ is a function that tells you the velocity of an object (measured in meters-per-second) as a function of the time t (measured in seconds). Remember that the integral is obtained by approximating the area between the graph of $v(t)$ and the t-axis by drawing a bunch of rectangles. So what units does the area of a rectangle have?

If we draw a rectangle of height v (measured in meters-per-second) and width Δt (measured in seconds), the product $v \cdot \Delta t$ has units

$$\frac{\text{meters}}{\text{seconds}} \cdot \text{seconds.}$$

The units of seconds cancel, leaving us with units of meters. In other words, the area under the graph of $v(t)$ naturally has units of meters—i.e., the integral must represent some sort of distance!

This makes sense: Velocity times time equals distance traveled. The rectangles we draw when computing Riemann sums represent our attempts at computing distance traveled by replacing $v(t)$ (where velocity could be changing all the time) with a bunch of functions that look constant (as though velocity were constant over the intervals Δt).

The upshot. If $v(t)$ is a velocity function and t is a time variable, then

$$\int_a^b v(t)\, dt$$

represents the *distance* traveled between time a and time b. Informally, *the integral of velocity (with respect to time) is distance.*

Remark 16.8.3. This makes sense if you think of "integrals" (i.e., antiderivatives) as the reverse operation of "derivatives." You take the derivative of position to obtain velocity; so the antiderivative of velocity is position.

16.8.3 Density and mass

Here is another common situation where integrals come up. Let's suppose that you have a rod, but the rod is made up of a non-uniform material. We'll be concerned with the density of the rod, which is a measure of how much mass is contained in a particular portion of the rod. (For example, the rod may be made of heavier material at one end than at the other end.)

Then we may be given a function $\rho(l)$ which tells us the density of the rod l meters away from one end of the rod.[6] We'll be interested in density as measured in "kilograms per meter."[7] If you think about it for a moment: If you know the density of a rod in terms of kilograms per meter, then if you multiply this density by the length of the rod (meters), you should obtain the mass of the rod itself. This would be very much true if the rod had uniform/constant density, but in our situation, ρ changes with respect to l.

But what if we compute the integral

$$\int_a^b \rho(l)\, dl\,?$$

Again approximating this integral using rectangles (using a Riemann sum), we see that each rectangle's area represents a quantity in units of

$$\frac{\text{kilograms}}{\text{meters}} \cdot \text{meters}.$$

In other words, each rectangle's area has units of kilograms. (Just as with velocity, each rectangle represents an approximation where we pretend that density is constant over the width of the rectangle.) Adding all the areas of these rectangles up, we obtain something in units of kilograms—and it is the mass of the portion of the rod living over the interval $[a, b]$.

Upshot. The integral of density (with respect to length) is mass.

Remark 16.8.4. In fact, this kind of thinking works for all kinds of situations when you replace density with concentration per unit length. (You can think of density as the concentration of "mass" per unit length.) For example, if a straight piece of string was soaked with a chemical, and the concentration of the chemical is changing along its length, the integral of this concentration (with respect to length) would represent the total amount of chemical soaked into the string.

[6] ρ is a Greek letter, read "rho." ρ is the lower-case form. P is the capital form. I don't know why, but it is common to denote density by this Greek letter.

[7] In real life, we most often measure density in kilograms per cubic meter, but we only care about how density changes as we move along the length of our rod, so we'll use kilograms per meter.

Remark 16.8.5. You might be a little dissatisfied that we've only talked about rods and strings. In real life, we might be interested in concentrations of chemicals measured over swaths of land (i.e., an actual region, now just a string), or in density of substances with actual volume. Indeed, you can take integrals over things like swaths of land and volumes of things, but we won't talk about that in this class. You'll learn how to do such things if you take a "multivariable calculus" class, which is sometimes called "Calculus III." (Also, a dirty secret: I think that Calculus III can be taken successfully without ever taking Calculus II.)

16.8.4 Getting used to wonky units

Sometimes, we have to get used to wonkier units.

For example, let's say $P(t)$ represents the population (of a country, say) at time t. So P is measured in units of persons; and let's say we measure t in years. Then what would

$$\int_a^b P(t)\,dt$$

represent? Whatever it represents, it will have units of "person-years." (You can think of this as "Person×Years.") If you aren't used to this kind of thing, these units can seem quite strange.

But they can be quite useful. For example, let's say that on average, a person in our country consumes 25MWh (megawatt-hours)[8] of natural gas per year in energy usage. (This is about the average for United States as of 2015.) In other words, our country consumes about 25MWh's worth of natural gas per person per year.

So if we multiply the above integral by 25, we obtain something in units of

$$\frac{\text{MWh}}{\text{person-year}} \times \text{person-year}.$$

In other words, we'll determine the total amount of natural gas (in terms of megawatt-hours) that we as a country consumed between time a and time b.

16.8.5 Work examples

In physics, *work* is defined as "force times distance." For example, let's say you want to lift a box weighing 10 kilograms 5 meters. It turns out that the force of gravity

[8]Megawatt-hours is a common unit of measuring energy. As a point of comparison, in the US, as of 2014, it's estimated that the average person uses about 13MWh of electricity per year. (This is more than quadruple the global average electricity use.)

is always acting on that box, and this force can be computed to be 98 Newtons. If you want to lift the box 5 meters, then the work required to that is

$$98 \text{ Newtons} \times 5 \text{ meters} = 490 \text{ Newton-meters}.$$

In general, if you want to lift an object weighing m kilograms a height of h meters, the work required is

$$\text{force times distance} = m \times 9.8 \text{ Newtons} \times h \text{ meters} = 9.8mh \text{ Newton-meters}.$$

This should gice some intuition about the word work—the more something weighs, of course it takes more "work" to lift it; and the farther up you want to lift it, of course it takes more "work."

Remark 16.8.6. It turns out that Newton-meters is also the unit for energy; in other words, work is measured in energy units. And this is actually the real use of work in physics.

As an example, let's suppose that you lift up a 10-kilogram box by 5 meters, so you do 490 Newton-meters of work (as we showed above). Now let's say you drop the box. It turns out that, when the box hits the ground again, it has a kinetic energy of exactly 490 Newton-meters (ignoring air resistance for the discussion). Kinetic energy is defined as one-half of mass times velocity-squared; it is the energy of a moving object.

There are many situations where the force in question is not constant.

Example 16.8.7. For an ideal spring, there is a number k so that when the spring is stretched x meters from its natural state, you need to exert a force of kx Newtons to maintain the spring's stretched state. This is called Hooke's Law; you usually learn about it in a course in physics. The number k is called the spring constant.

How much work does it take to stretch a spring from 0 to 0.2 meters from its natural state?

Well, at any given moment when the spring is stretched x meters, one must exert kx Newtons of force. To stretch this a further Δx meters, the work to do that would be $kx \times \Delta x$.

So, if we were to approximate this process using tinier and tinier intervals of stretching, we would end up with an expression like

$$\sum_{i=1}^{n} kx_i \Delta x$$

where we are using n tiny intervals of stretch, $x_1 = 0 + \Delta x$ and $x_n = 0.2$. Of course, by definition, if we take $n \to \infty$, we find the integral:

$$\int_0^{0.2} kx\,dx.$$

This integral becomes

$$\frac{1}{2}kx^2\Big|_0^{0.2}.$$

16.8.6 Pandemic examples

The most common graph you've seen on the news during the pandemic has to do with "new infections." If you model this bar graph using a function $f(t)$, then $f(t)$ can be measured in units of "persons per day" (measuring the number of people testing positive per day) and t can be measured in units. Then the integral

$$\int_a^b f(t)\,dt$$

has units of (persons per day)×(days); that is, units of "persons." It measures the number of people who tested positive between time a and time b.[9]

Similarly, if $g(t)$ is a function modeling the number of new hospitalizations per day, the integral $\int_a^b g(t)\,dt$ measures the total number of hospitalizations between time a and time b. Sometimes, we are more interested in $g(t)$ than in the integral, because hospitals may be most afraid of a "rush" of new patients—that is, a hospital may be able to handle 10,000 hospitalizations over the course of a year, but certainly not if they all come at once.

Regardless, let's let $G(t)$ represent the total number of people in the hospital for COVID-19 at time t. Then $\int_a^b G(t)\,dt$ is measured in person-days. This is a potentially useful measure of how much of our hospital resources COVID-19 is occupying. After all, this integral takes into account the fact that a patient who spends 20 days in the hospital may require more hospital resources than a patient who spends only 1 day in the hospital.

16.9 Basic laws of integration

Before we get to u substitution, here are some basic properties about integrals.

[9]This "number of people who tested positive" is a bit misleading; it is possible that some people test positive multiple times.

(I) **You can concatenate intervals.**

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx. \tag{16.9.1}$$

You actually used the above fact last time: To find the area of f over an interval $[a, c]$, you can divide the interval into two pieces–$[a, b]$ and $[b, c]$—and computed the area over each of those smaller intervals, then add the result. Note that b doesn't need to be a midpoint or anything; it's just any point between a and c. Here, for example, is a consequence of the above fact:

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx. \tag{16.9.2}$$

(II) **Integrals scale.** Here is another fact. If m is any real number, we have

$$\int_a^b mf(x)dx = m\int_a^b f(x)dx. \tag{16.9.3}$$

That is, area scales (by m) if you scale f (by m). Intuitively, if you think of $f(x)$ as describing the height of a curvy fence at position x, if you make the fence m times taller everywhere, the area also grows by m times. Here, m is any real number. It could be zero or negative or positive.

(III) **If you add the integrands, you add the integrals.** Another fact that will be useful is

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{16.9.4}$$

Informally, if you have something of height $f(x)$ at x, then you add on a height of $g(x)$ at every x, then the area of the resulting figure is obtained by adding the areas given by f and given by g.

(IV) **Reverse, reverse.** The final fact that will be useful is

$$\int_a^b f(x)dx = \int_b^a -f(x)dx \tag{16.9.5}$$

This is is a somewhat strange rule, but it will come up. So far, we've always had $\int_a^b$ consist of numbers such that $a < b$; but as we compute, we will sometimes end up with $\int_b^a$ with $b > a$. If this happens, you just reverse the sign of f and reverse the roles of b and a. If you like, $\int_a^b$ tells you to take the area "from a to b," which usually has meant "from left to right." If the integral tells us to reverse directions by going from right to left (e.g., from b to a), we think of the contributions from f negatively.

That's it. Three rules. As a student, you might feel good that there's "less to memorize," but actually, having fewer rules means you have fewer guideposts to rely on, so you need to be more creative about how to compute integrals!

16.10 Some notation

For today you've practiced taking integrals. For example, to compute

$$\int_1^4 \frac{1}{x} dx,$$

you must

1. find an antiderivative F to the integrand, and then
2. compute $F(b) - F(a)$ (which in this case if $F(4) - F(1)$).

We know that $\ln(x)$ has derivative given by $1/x$, so we can take $F(x) = \ln x$. We thus find

$$\int_1^4 \frac{1}{x} dx = \ln(4) - \ln(1) \tag{16.10.1}$$
$$= \ln(4) - 0 \tag{16.10.2}$$
$$= \ln 4. \tag{16.10.3}$$

Let me introduce the following:

Notation 16.10.1. We write

$$F(x)\Big|_a^b$$

to mean

$$F(b) - F(a).$$

In other words,

$$F(x)\Big|_a^b = F(b) - F(a).$$

Example 16.10.2. So for example, the above work could have been written

$$\int_1^4 \frac{1}{x} dx = \ln(x)\Big|_1^4 \tag{16.10.4}$$
$$= \ln(4) - \ln(1) \tag{16.10.5}$$
$$= \ln(4) - 0 \tag{16.10.6}$$
$$= \ln 4. \tag{16.10.7}$$

As usual, this notation is meant to *help* you. I promise you'll begin to use this notation in no time, because you'll always first the antiderivative first, and then you'll plug in b and a. This "vertical bar" notation will help you keep track of those steps as you write.

16.11 More terms

Here are some terms that are commonly uses in calculus classes; I am thus obligated to tell you about them, so that you are note excluded from the culture of calculus.

Definition 16.11.1. The *indefinite integral* is an old terminology that survives mostly in calculus textbooks. The notation

$$\int f(x)dx$$

is called an "indefinite" integral because the integral does not specify *where* we are integrating—i.e., there is no a and no b.

A lot of calculus textbooks ask you to "solve the indefinite integral." I am not a fan—at all—of this terminology. Regardless, because it is so prevalent in calculus culture, I have to tell you about it to prevent you from experiencing confusion or frustration should some other calculus professor torture you with this old terminology.

To "solve an indefinite integral" means to "find all antiderivatives of $f(x)$." (You saw last time—via the Fundamental Theorem of Calculus (FTC)—why finding an antiderivative helps you find the integral.)

If F is an antderivative of $f(x)$, it is customary to write the following as a "correct answer" to the calculus textbook's problem of finding an indefinite integral:

$$\int f(x)dx = F(x) + C.$$

The $+C$ is a perennially confusing notation. Let me explain it. Recall from the lecture about the mean value theorem that we discovered the following fact: If G and F have the same derivative (meaning $F' = G'$) then $F - G$ must be a constant function. For example, it could be that $F - G = \pi$ or $F - G = 0$ or $F - G = 3$. The important part is that it could be any constant. So if F is an antiderivative, *every* other antiderivative can be expressed as $F+$ something, where something is a constant. that's what the $+C$ above means. The C stands for "constant," and the notation $\int f(x)dx = F(x) + C$ is a lazy way of saying "any antiderviative of f can be obtained from $F(x)$ by adding some constant C."

Example 16.11.2. The indefinite integral $\int \sin(x)dx$ is given by

$$-\cos(x) + C.$$

I really dislike the language of "indefinite integral," because we already have a perfectly good word for the underlying idea one is exploring: Antiderivative. The "$+C$" warning really just gives you a heads up that a lot of integral tables and encyclopedia articles have $+C$ because an antiderivative really could be obtained from F by adding some constant, and a particular constant may be very important for a particular engineering problem. But if some calculus person ever asks you to find an "indefinite" integral, just find the antiderivative and write "$+C$." And know that the only reason you're writing $+C$ is to indicate that any other antiderivative can be obtained by adding a constant to the one you found.

16.12 Preparation for next time

For next time, I expect you to be able to find antiderivatives of simple functions (Exercise 16.6.1). I expect you to be able to also compute areas using the fundamental theorem of calculus (Exercise 16.6.3). Finally, I also expect you to be able to tell me what units an integral should have given the units of the integrand and the input variable.