Lab exercises involving integration of simple functions

Exercise 16.11.1. For each of the functions below, find an antiderivative. Add 17 to your antiderivative, and verify that the result is still an antiderivative for the original function.

(a) $4x$

(b) $7x$

(c) 3

(d) π

(e) e

(f) e^x

(g) $3x + 7$

(h) $3x^2$

(i) x^2

(j) $\sqrt{x}$

(k) $9\sin(x)$

(l) $\cos(x)$

(m) $\frac{1}{x^3}$

(n) $\frac{1}{x^2}$

(o) $\cos(x) - 9x^2$

(p) $\frac{1}{x}$

(q) $\frac{2}{x^2} + \frac{3}{x} + x^{1/3}$

(r) $\frac{1}{\cos^2(x)}$

Possible solutions to Exercise 16.11.1.

(a) $2x^2$

(b) $\frac{7}{2}x^2$

(c) $3x$

(d) πx

(e) ex

(f) e^x

(g) $\frac{3}{2}x^2 + 7x$

(h) x^3

(i) $\frac{1}{3}x^3$

(j) $\frac{2}{3}x^{3/2}$

(k) $-9\cos(x)$

(l) $\sin(x)$

(m) $\frac{-1}{2x^2}$

(n) $\frac{-1}{x}$

(o) $\sin(x) - 3x^3$

(p) $\ln(x)$

(q) $\frac{-2}{x} + 3\ln x + \frac{3}{4}x^{4/3}$

(r) $\tan(x)$

□

Exercise 16.11.2. For each of the functions below, find an antiderivative.

(a) $x^3 + x - 9$

(b) $\sin(x) - \cos(x) + e^x$

(c) $\frac{7}{x} + \frac{1}{x}^2$

(d) $3 + \sin(x) - \frac{1}{x}$

(e) $e^x + \cos(x) + \frac{1}{x}$

(f) $3x^3 - 9e^x + 4\sin(x)$

(g) $\frac{1}{x^4} + x^4 + x^{3/5}$

(h) $\sin(3x)$

Possible solutions to Exercise 16.11.2.

(a) $\frac{1}{4}x^4 + \frac{1}{2}x^2 - 9x$

(b) $-\cos(x) - \sin(x) + e^x$

(c) $7\ln(x) + \frac{-1}{x}$

(d) $3x - \cos(x) - \ln(x)$

(e) $e^x + \sin(x) + \ln(x)$

(f) $\frac{3}{4}x^4 - 9e^x - 4\cos(x)$

(g) $\frac{-1}{3x^4} + \frac{1}{5}x^5 + \frac{5}{8}x^{8/5}$

(h) $\frac{-1}{3}\cos(3x)$

□

Exercise 16.11.3. Using the fundamental theorem of calculus, compute the following integrals.

(a) $\int_1^4 3x\,dx$

(b) $\int_{-1}^1 x^3\,dx$

(c) $\int_1^4 x^3\,dx$

(d) $\int_{-2}^2 x^3 + 3x^2\,dx$

(e) $\int_{\pi/4}^{\pi/2} \sin(x)\,dx$

(f) $\int_{\pi/4}^{\pi/2} \cos(x)\,dx$

(g) $\int_{\ln 2}^{\ln 6} 3e^x + 9\,dx$

(h) $\int_1^7 \frac{1}{x} + x^2\,dx$

Possible solutions to Exercise 16.11.3.

(a)

$$\begin{aligned}\int_1^4 3x\,dx &= \frac{3}{2}x^2\Big|_1^4\\ &= \frac{3}{2}4^2 - \frac{3}{2}1^2\\ &= \frac{45}{2}\end{aligned}$$

(b) 0

(c)

$$\begin{aligned}\int_1^4 x^3\,dx &= \frac{1}{4}x^4\Big|_1^4\\ &= \frac{1}{4}4^4 - \frac{1}{4}1^4\\ &= \frac{255}{4}\end{aligned}$$

(d)

$$\begin{aligned}\int_{-2}^2 x^3 + 3x^2\,dx &= \frac{1}{4}x^4 + x^3\Big|_{-2}^2\\ &= \frac{1}{4}2^4 + 2^3 - (\frac{1}{4}(-2)^4 + (-2)^3)\\ &= 16\end{aligned}$$

(e)

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \sin(x)\,dx &= -\cos(x)\Big|_{\pi/4}^{\pi/2}\\ &= -\cos(\pi/2) - (-\cos(\pi/4))\\ &= 0 + \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

(f)

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \cos(x)\,dx &= \sin(x)\Big|_{\pi/4}^{\pi/2}\\ &= \sin(\pi/2) - \sin(\pi/4)\\ &= 1 - \frac{\sqrt{2}}{2}\\ &= \frac{2-\sqrt{2}}{2}\end{aligned}$$

(g)

$$\begin{aligned}\int_{\ln 2}^{\ln 6} 3e^x + 9\,dx &= 3e^x + 9x\Big|_{\ln 2}^{\ln 6}\\ &= 3e^{\ln 6} + 9\ln 6 - (3e^{\ln 2} + 9\ln 2)\\ &= 18 + 9\ln 6 - (6 + 9\ln 2)\\ &= 12 + 9(\ln 6 - \ln 2)\\ &= 12 + 9\ln(6/2)\\ &= 12 + 9\ln(3)\end{aligned}$$

(h)

$$\begin{aligned}\int_1^7 \frac{1}{x} + x^2\,dx &= \ln(x) + \frac{1}{3}x^3\Big|_1^7\\ &= \ln(7) + \frac{1}{3}7^3 - (\ln(1) + \frac{1}{3}1^3)\\ &= \ln(7) + \frac{1}{3}7^3 - (0 + \frac{1}{3}1^3)\\ &= \ln(7) + \frac{342}{3}\\ &= \ln(7) + 114\end{aligned}$$

□

Exercise 16.11.4. Using the fundamental theorem of calculus, compute signed area between the following functions and the x-axis, between the vertical lines indicated.

(a) $f(x) = 3x$, between the lines $x = 1$ and $x = 4$.

(b) $g(x) = x^2$, between the lines $x = -1$ and $x = 1$.

(c) $h(x) = x^3$, between the lines $x = 1$ and $x = 4$.

(d) $f(x) = x^3 + 3x^2$, between the lines $x = -2$ and $x = 2$.

(e) $f(x) = \sin(x)$, between the lines $x = \pi/4$ and $x = \pi/2$.

(f) $g(x) = \cos(x)$, between the lines $x = \pi/4$ and $x = \pi/2$.

(g) $h(x) = 3e^x + 9$, between the lines $x = \ln 2$ and $x = \ln 6$.

(h) $f(x) = \frac{1}{x} + x^2$, between the lines $x = 1$ and $x = 7$.

Possible solutions to Exercise 16.11.4. Same as solutions to Exercise 16.11.3. □

Exercise 16.11.5. It is known that a falling ball (in the absence of air resistance) has vertical upward velocity at time t given by

$$v(t) = v_0 - 9.8t$$

where v_0 is the speed of the ball at time t, t is measured in seconds, and $v(t)$ is measured in meters per second.

(a) What are the units of $v'(t)$?

(b) What are the units of $\int_a^b v(t)\,dt$?

(c) Suppose that a ball is dropped with initial velocity $v_0 = 0$ from a height of 100 meters. What is the ball's height after 1 second? After 2 seconds? After 3 seconds?

(d) Suppose that a ball is thrown from a height of 3 meters with initial velocity $v_0 = 2$. How high up does the ball end up after 1 second? After 2 seconds? After 3 seconds?

Possible solutions to Exercise 16.11.5. (a) Meters per second per second (also known as meters per seconds-squared).

(b) Meters.

(c) $v(t) = -9.8t$. The following integrals tell us how far the ball moves:

$$\int_0^1 -9.8t\,dt = -4.9(1)^2 - (-4.9(0)^2) = -4.9.$$

$$\int_0^2 -9.8t\,dt = -4.9(2)^2 - (-4.9(0)^2) = -4.9 \times 4 = -19.6$$

$$\int_0^3 -9.8t\,dt = -4.9(3)^2 - (-4.9(0)^2) = -4.9 \times 9 = -44.1.$$

Because the ball starts from a height of 100 meters, the height after 1 second is 100 - 4.9 = 95.1, after 2 seconds is 100 - 19.6 = 80.4, and after 3 seconds is 100 - 44.1 = 55.9 (all in meters). By the way, ignoring air resistance, these numbers are all accurate. If an object is dropped and it hits the ground after 3 seconds, the object was 44.1 meters up from the ground. Note also that between $t = 1$ and $t = 2$, the ball travels a different amount of height than say between $t = 2$ and $t = 3$, or $t = 0$ and $t = 1$. This is because falling objects do not have a constant speed as they fall; they accelerate toward the ground due to gravity.

(d) The following integrals tell us how far the ball travels vertically:

$$\int_0^1 2 - 9.8t\,dt = 2t - 4.9t^2\Big|_0^1 = 2(1) - 4.9(1)^2 - (2(0) - 4.9(0)^2) = 2 - 4.9 = -2.9$$

$$\int_0^2 2 - 9.8t\,dt = 2t - 4.9t^2\Big|_0^2 = 2(2) - 4.9(2)^2 - (2(0) - 4.9(0)^2) = 4 - 19.6 = -15.6$$

$$\int_0^3 2 - 9.8t\,dt = 2t - 4.9t^2\Big|_0^3 = 2(3) - 4.9(3)^2 - (2(0) - 4.9(0)^2) = 6 - 44.1 = -38.1$$

So if the ball started at a height of 3 meters, it ends up at 0.1 meters, -12.6 meters, and -35.1 meters above the ground at times $t = 1, 2, 3$ seconds, respectively. (Notice that even when thrown upward at 2 meters per second from a height of 3 meters, after 1 second, the ball is nearly at the ground already!)

□

Exercise 16.11.6. One can measure the rate at which water is flowing in and out of a bay of water on some planet. We model the flow rate by the function

$$r(t) = 13\sin(t)$$

where $r(t)$ is in units of acre-feet per hour, and t is measured in hours. A positive flow rate means water is entering the bay, while a negative flow rate means water is escaping the bay.

1. What are the units of $\int_a^b r(t)\,dt$?

2. What are the units of $r'(t)$?

3. How much water has escaped or entered the bay between $a = \pi/4$ hours and $b = \pi/2$ hours?