Lecture 15

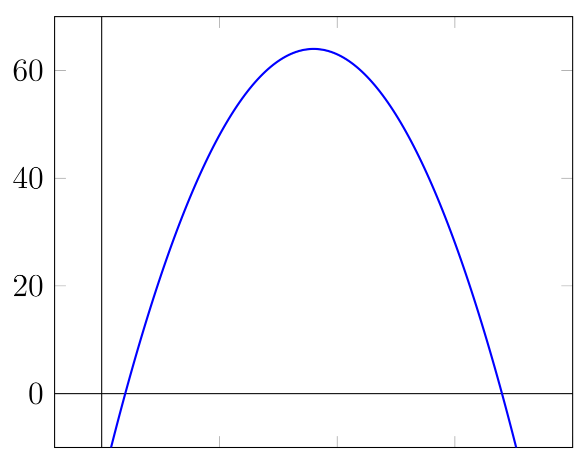
Riemann sums

The three biggest ideas in calculus are (1) Limits, (2) Derivatives, and (3) Integrals.

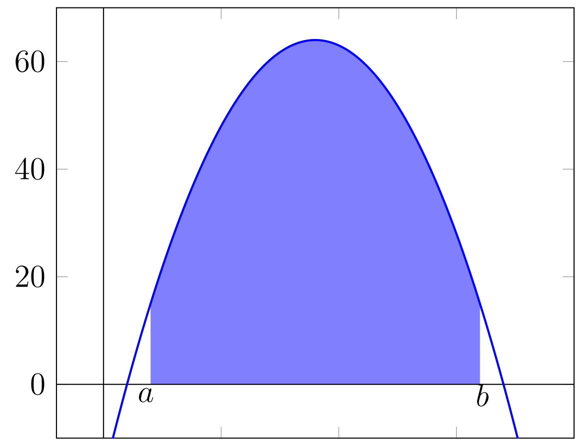
For the next few weeks, we'll be talking about integrals. If derivatives are about slope, then integrals are about area.

15.1 Areas

Below is the graph of a function $f(x)$.

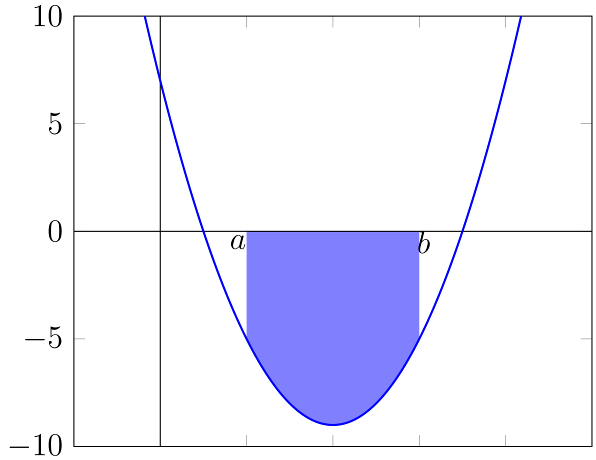


Let's choose two numbers a and b so that $a < b$, and let's fill in the region between the graph of f and the x-axis contained between a and b:

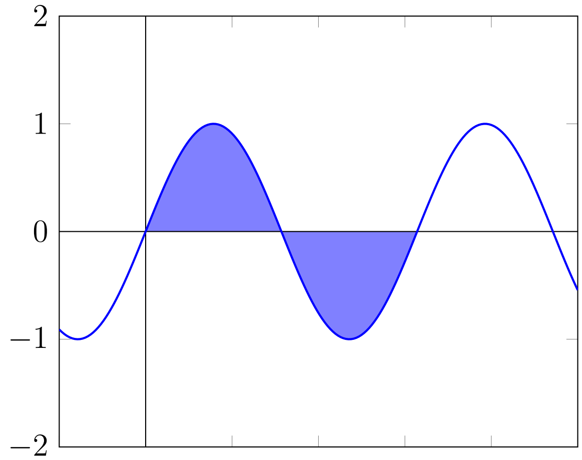


Can we calculate the *area* of that region?

Convention 15.1.1 (Signed area)**.** We will be interested in the *signed* area of a region. This means that if the graph of f is below the x axis, we will consider the corresponding area to be *negative*. For example, we will treat the region below as having negative area:

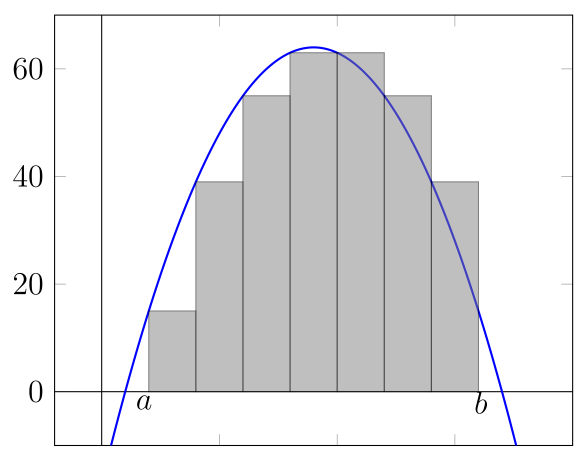


As another example, the next region will have a total area of **zero**, because the positive area cancels the negative area:



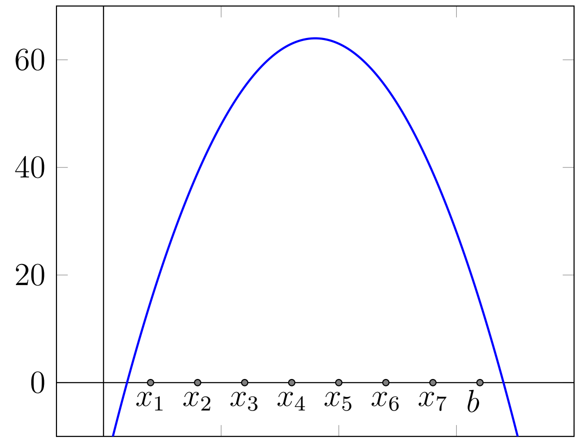
15.2 Approximating areas

Here is a maddeningly naive way to approximate area: We try to approximate the region of interest by rectangles. Moreover, we will approximate it with rectangles in a very simple way, as follows:

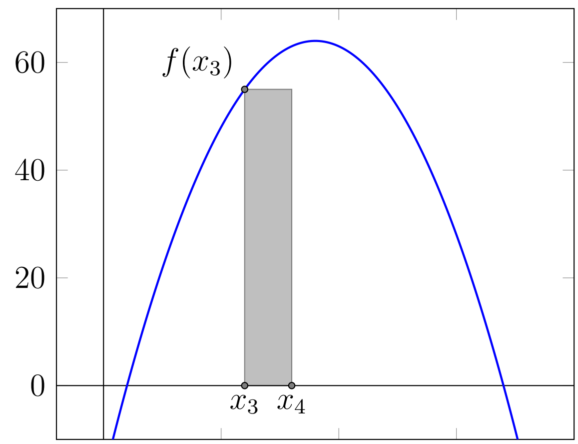


(15.2.1)

Here is what we have done: We divided the interval from a to b into 7 equal pieces. This gives us 7 numbers along the x axis, which we will call $x_1, x_2, \ldots, x_6, x_7$.



Then for every number x_i, we drew a rectangle of height $f(x_i)$, with width going from x_i to x_{i+1}. For example, here is the rectangle of height $f(x_3)$:



And doing this for every x_i (that is, for $x_1, x_2, \ldots, x_6, x_7$) gives us the picture (15.2.1) from before.

Now, if we add up the areas of all the rectangles, we get a number. What are the areas? Well, the area of the rectangle corresponding to x_3 is given by

$$\text{height} \times \text{width} = f(x_3) \times (x_4 - x_3).$$

But we chose the numbers x_i so that they were evenly spaced between b and a, and there were exactly 7 intervals. So $x_4 - x_3 = (b-a)/7$. In other words,

$$\text{Area of 3rd rectangle} = f(x_3) \times \frac{b-a}{7}.$$

And in general, if we wanted the ith rectangle, we get

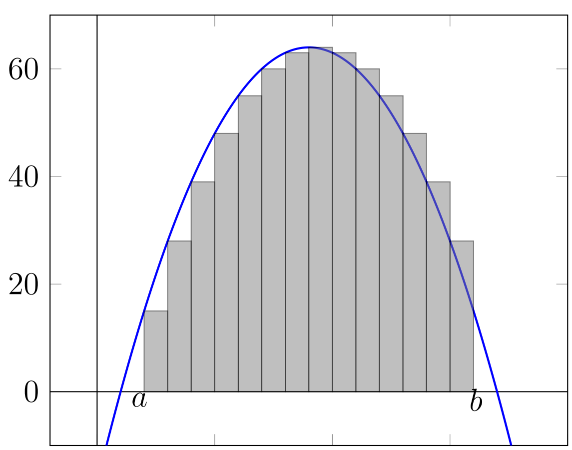
$$\text{Area of } i\text{th rectangle} = f(x_i) \times \frac{b-a}{7}.$$

Adding everything together, we get

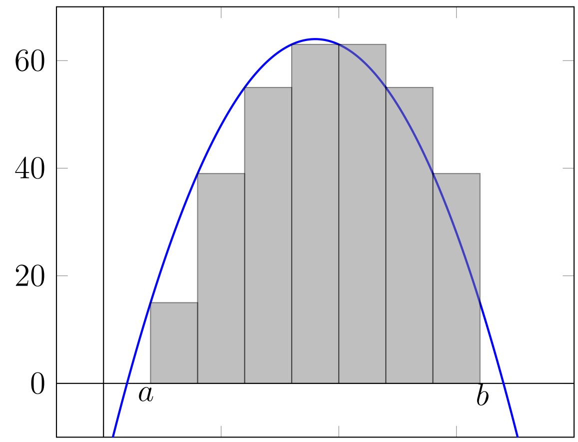
$$\begin{aligned}
&\text{Sum of areas of all rectangles} && (15.2.2)\\
&= f(x_1) \times \frac{b-a}{7} + f(x_2) \times \frac{b-a}{7} + f(x_3) \times \frac{b-a}{7} + f(x_4) \times \frac{b-a}{7} + f(x_5) \times \frac{b-a}{7} + f(x_6) \times \frac{b-a}{7} + \\
&= [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)]\frac{b-a}{7}. && (15.2.3)
\end{aligned}$$

So this is one way to *approximate* the area of the region.

If we want to get a better approximation, we could take, for example, *fourteen* rectangles instead of just seven.



Let's compare this to the previous picture with 7 rectangles:



It does, indeed, seem like having more rectangles gives a better approximation to the area.

Remark 15.2.1. You can tell that the rectangles do not give a perfect approximation. For example, above, the rectangles on the left seem to *under*estimate the area under the graph of f—this is because there is a gap between the graph of f and the top of the rectangles. And, for the rectangles on the right, it seems that the areas of the rectangles *over*estimate the area of under the graph of f, as there is an excess of shaded grey region.

15.2.1 Exercises

Exercise 15.2.2. (a) Write a sum to approximate the signed area between the x-axis and the graph of the function $f(x) = x^2$ between $x = 2$ and $x = 5$, using *three* rectangles of equal width.

(b) Do the same, using *six* rectangles of equal width.

(c) Do the same, using *seven* rectangles of equal width.
If you have a calculator, write out the values of the approximation you find.

One possible solution. (a) If we divide the interval from $x = 2$ to $x = 5$ into three equal parts, we obtain the three intervals

$$[2, 3], \qquad [3, 4], \qquad [4, 5].$$

In all of our pictures so far, we have drawn rectangles above these intervals whose heights are given by the value of f at the *lefthand* endpoint of these intervals. So we have three rectangles whose signed areas are given by

$$f(2) \times 1, \qquad f(3) \times 1, \qquad f(4) \times 1$$

where 1 is the width of the three intervals. The sum of the these three areas is given by

$$\begin{aligned} f(2) \times 1 + f(3) \times 1 + f(4) \times 1 &= 2^2 \times 1 + 3^2 \times 1 + 4^2 \times 1 \\ &= 4 + 9 + 16 \\ &= 29. \end{aligned} \tag{15.2.4}$$

(b) Dividing the interval from 2 to 5 into six equal parts, we obtain the six intervals

$$[2, 2.5], \qquad [2.5, 3], \qquad [3, 3.5], \qquad [3.5, 4], \qquad [4, 4.5], \qquad [4.5, 5].$$

The rectangles above these intervals, if we give them heights equal to the value of f at the lefthand endpoint of each interval, have areas

$$f(2)\times 0.5, \qquad f(2.5)\times 0.5, \qquad f(3)\times 0.5, \qquad f(3.5)\times 0.5, \qquad f(4)\times 0.5, \qquad f(4.5)\times 0.5.$$

Adding these up, we get

$$\begin{aligned} &f(2) \times 0.5 + f(2.5) \times 0.5 + f(3) \times 0.5, + f(3.5) \times 0.5, + f(4) \times 0.5, + f(4.5) \times 0.5 \\ &= (2^2 + (2.5)^2 + 3^2 + 3.5^2 + 4^2 + 4.5^2) \times 0.5 \\ &= (\frac{16}{4} + \frac{25}{4} + \frac{36}{4} + \frac{49}{4} + \frac{64}{4} + \frac{81}{4}) \times 0.5 \\ &= \frac{1}{8}(16 + 25 + 36 + 49 + 64 + 81) \\ &= 271/8 \\ &= 33.875. \end{aligned} \tag{15.2.5}$$

(c) Dividing the interval from 2 to 5 into 7 equal intervals, we know that each interval should have width $(5 - 2)/7 = 3/7$. So the seven intervals are given as

$$[2, 2+\frac{3}{7}], \qquad [2+\frac{3}{7}, 2+2\frac{3}{7}], \qquad [2+2\frac{3}{7}, 2+3\frac{3}{7}], \qquad [2+3\frac{3}{7}, 2+4\frac{3}{7}], \qquad [2+4\frac{3}{7}, 2+5\frac{3}{7}], \qquad [2+5\frac{3}{7}, 2-$$

Each rectangle has width 3/7, and heights given by the lefthand endpoints of each interval; so adding up the areas, we find

$$\left(f(2) + f(2+\frac{3}{7}) + f(2+2\frac{3}{7}) + f(2+3\frac{3}{7}) + f(2+4\frac{3}{7}) + f(2+5\frac{3}{7}) + f(2+6\frac{3}{7})\right)\times\frac{3}{7}.$$

It turns out this is equal to roughly 34.6.

(It also turns out that the actual area under this function, from $x = 2$ to $x = 5$, is given by 39.) □

Exercise 15.2.3. Write a sum to approximate the signed area between the x-axis and the graph of the function $f(x) = x + \sin(x)$ between $x = 1$ and $x = 2$, using *four* rectangles of equal width.

If you have a calculator, write out the values of the approximation you find.

15.3 Summation notation (Sigma notation)

At this point, let's set the area problem aside for a moment and look at the (15.2.3) equation above. **This is no way to live life.** Our summation in (15.2.3) has seven terms (which can seem like a lot) and it literally does *not* fit on the page. We need a better way to encapsulate this summation.

So we will introduce something called *summation notation*, or Σ *notation*. (The symbol Σ is read "sigma." It is a Greek letter, and Σ eventually evolved into S. In care you're curious, Σ is the upper-case, or capital form. The lower-case form of sigma is σ.)

Definition 15.3.1 (Summation notation, I)**.** Suppose you have 7 numbers

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7.$$

Then we will write their sum as

$$\sum_{i=1}^{7} a_i.$$

In other words, we *define*

$$\sum_{i=1}^{7} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7.$$

Note that we have given each number a name—for example, a_1 or a_2. If we were parents and we were naming our children, perhaps such names would be dehumanizing. But that's what we'll do to our numbers.

The symbol

$$\sum$$

means "we're summing things"—that is, "we're adding."

You may have seen in a previous class that the things we're adding together are called *summands*. That is, each of the $a_1, a_2, \ldots, a_7$ is a summand for our summation.

The superscript and subscript

$$\sum_{i=1}^{7}$$

means "we're going to be adding numbers *labeled* by integers from 1 through 7," and I am going to use i as a variable to stand in for these labels. In other words, whenever you see an i, you will eventually plug in an integer between 1 and 7.

Finally,

$$\sum_{i=7}^{7} a_i$$

specifies *what* we're adding—we're adding the numbers $a_1, \ldots, a_7$.

Exercise 15.3.2. Below are examples of summation notation. Write out the full summation encoded in the notation.

(a) $\sum_{i=1}^{3} i$

(b) $\sum_{n=1}^{4} n^2$

(c) $\sum_{i=3}^{7} (-1)^i$

(d) Let $f(x) = \sin(x)$. Write out $\sum_{j=2}^{5} f(j)$.

(e) Let $f(x) = \sin(x)$ and let $x_n = 3 + 0.1n$. Write out $\sum_{n=1}^{4} f(x_n)$.

(f) Let $f(x) = \sin(x)$ and let $x_n = 3 + 0.1n$. Write out $\sum_{n=0}^{4} f(x_n)(3.5 - 3)/5$.

Solutions.

(a) $\sum_{i=1}^{3} i = 1 + 2 + 3 = 6.$

(b) $\sum_{n=1}^{4} n^2 = 1 + (2)^2 + (3)^2 + (4)^2 = 1 + 4 + 9 + 16 = 30.$

(c) $\sum_{i=3}^{7} (-1)^i = (-1)^3 + (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 = -1 + 1 - 1 + 1 - 1 = -1.$

(d)

$$\begin{aligned}\sum_{j=2}^{5} f(j) &= f(2) + f(3) + f(4) + f(5) \\ &= \sin(2) + \sin(3) + \sin(4) + \sin(5). \qquad (15.3.1)\end{aligned}$$

(e)

$$\begin{aligned}\sum_{n=1}^{4} f(x_n) &= f(x_1) + f(x_2) + f(x_3) + f(x_4) \\ &= \sin(3.1) + \sin(3.2) + \sin(3.3) + \sin(3.4).\end{aligned}$$

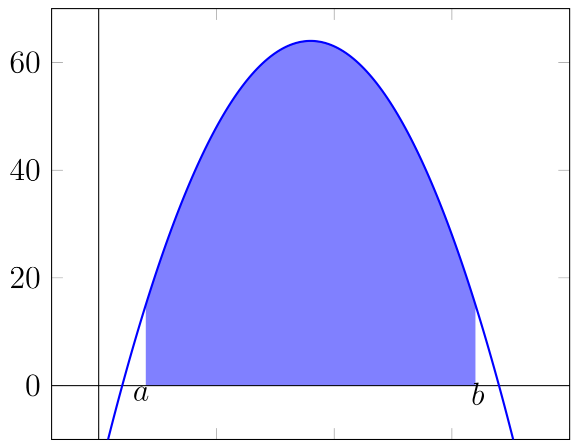
(f)

$$\begin{aligned}&\sum_{n=0}^{4} f(x_n)(3.5 - 3)/5 \qquad (15.3.2) \\ &= f(x_0) \cdot 0.5/5 + f(x_1) \cdot 0.5/5 + f(x_2) \cdot 0.5/5 + f(x_3) \cdot 0.5/5 + f(x_4) \cdot 0.5/5 \\ &= f(3) \cdot 0.1 + f(3.1) \cdot 0.1 + f(3.2) \cdot 0.1 + f(3.3) \cdot 0.1 + f(3.4) \cdot 0.1. \\ &= (f(3) + f(3.1) + f(3.2) + f(3.3) + f(3.4)) \cdot 0.1 \\ &= (\sin(3) + \sin(3.1) + \sin(3.2) + \sin(3.3) + \sin(3.4)) \cdot 0.1 \\ &\qquad (15.3.3)\end{aligned}$$

15.4 Riemann sums

A Riemann sum is a way to approximate the area between the graph of a function and the x axis.

(1) First, for whatever reason, we decide the interval over which we want to take the area. This is a choice of two numbers a and b satisfying $a < b$. We want to approximate the shaded region below:



(2) Second, we decide *how many* rectangles we want to draw. We will call this number n. **We will always draw n rectangles of equal width.** This is not strictly necessary, but we are following tradition and convention with this equal-width requirement.

Of course, if there are n rectangles of equal width, the width of each rectangle must then be

$$\Delta x = (b - a)/n.$$

I have used the symbol Δx; this is the change in the x coordinate that we experience when we travel along the width of a rectangle.

Thus, we can identify where the rectangles' widths begin and end: We have rectangles above the following intervals:

$$[a, a+\Delta x], \qquad [a+\Delta x, a+2\Delta x], \qquad [a+2\Delta x, a+3\Delta x], \qquad \ldots \qquad [a+(n-1)\Delta x, b]$$

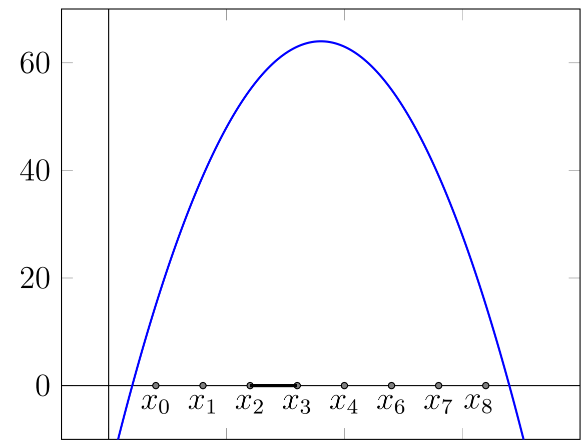
Because we will get tired of writing so many Δx and their multiples, we are going to be lazy (as usual). So instead of writing $a + k\Delta x$, we are going to let

$$x_k = a + k\Delta x.$$

This allows us to write the intervals above us

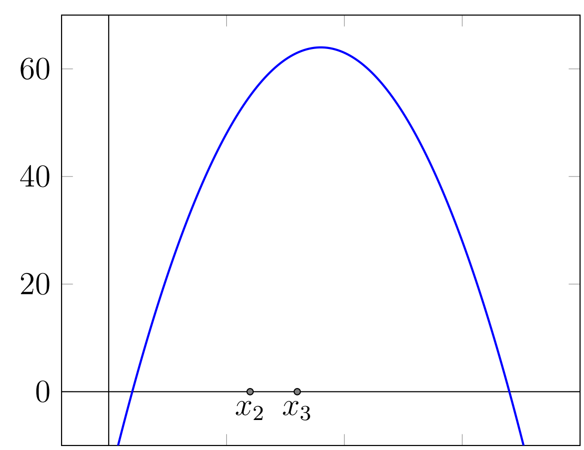
$$[x_0, x_1], \qquad [x_1, x_2], \qquad [x_2, x_3], \qquad [x_{n-1}, x_n].$$

This notational simplicity has a certain appeal to it, doesn't it? Below is what happens if we decide to divide $[a, b]$ into $n = 8$ intervals:



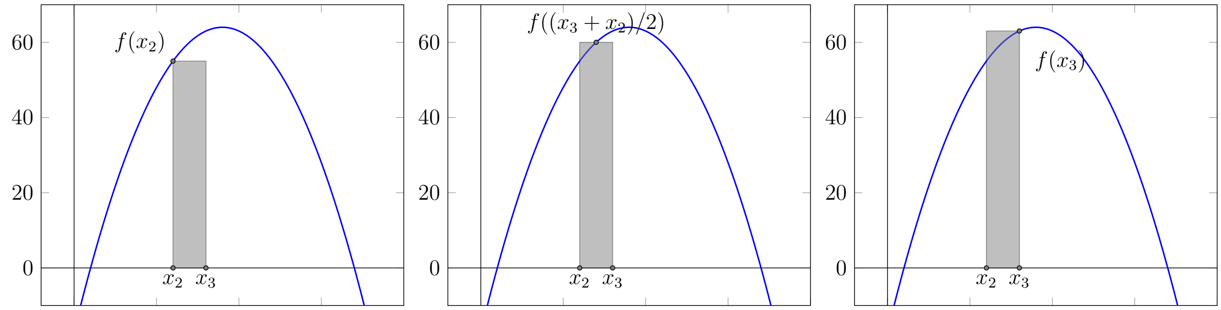
Note that $x_0 = a$ and $x_n = b$. Drawn thick is the interval from x_2 to x_3.

Third, we must decide on the *heights* of our rectangles. Again, in principle, there are many ways to choose the heights. For example, suppose somebody told you that they wanted to find a rectangle of some height to draw above the indicated interval of width Δx:



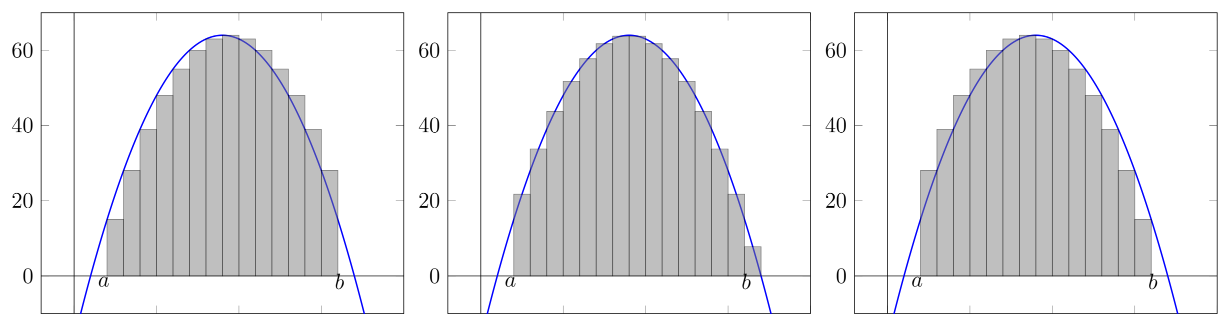
(15.4.1)

Here are three natural choices:



- In the lefthand image, we have chosen the rectangle to have height given by $f(x_2)$ (so that the left edge of the rectangle hits the graph of f). We will call this the **lefthand rule** for choosing the height of the rectangle.
- In the middle image, we have chosen the rectangle to have height given by the value of f at the midpoint between x_2 and x_3. We will call this the **midpoint rule** for choosing the height of a rectangle.
- In the righthand image, we have chosen the rectangle to have height given by $f(x_3)$ (so that the right edge hits the graph). We will call this the **righthand** rule for choosing the height of a rectnagle.

Any of these choices is fine, and just to make things simple, once we have chosen a rule, we will stick with it for the entire approximation. Here are drawings of the rectangles we obtain by using the three different rules to approximate the area underneath the graph of f:



15.5 Preparation for next time

For next time, I expect you to be able to do the following:

(a) Set up a Riemann sum to approximate the area under the parabola $f(x) = 9-x^2$ along the interval $[1, 7]$ using 3 rectangles. (Set up means to write out the summation both using Σ notation and usual addition; you do not need to then evaluate the sum to obtain a number.) You should be able to do this using the lefthand, the righthand, and the midpoint rules.

(b) Set up a Riemann sum to approximate the area under the sine curve $f(x) = \sin(x)$ along the interval $[0.1, 0.5]$ using 4 rectangles. You should be able to do this using the lefthand, righthand, and midpoint rules.

(c) Set up a Riemann sum to approximate the area under the curve $f(x) = \ln x$ along the interval $[1, 4]$ using 5 rectangles. You should be able to do this using the lefthand, righthand, and midpoint rules.

(d) Set up a Riemann sum to approximate the area under the curve $f(x)$ along the interval $[a, b]$ using n rectangles. You should be able to do this using the lefthand, righthand, and midpoint rules. (You may leave the answer to this problem in Σ notation.)

I will do the third example out to show you how it's done.

First, our interval $[a, b]$ is given by $[1, 4]$. (In other words, $a = 1$ and $b = 4$ in terms of our previous notation.) And, we want to use 5 rectangles. So each rectangle will have width $(4 - 1)/5 = 3/5$. So we will have a total of six x values (these are the x coordinates for the corners of the rectangles):

$$x_0 = 1, \qquad x_1 = 1 + 3/5 = 8/5, \qquad x_2 = 1 + 2 \times 3/5 = 11/5, \qquad \ldots, \qquad x_5 = 4.$$

Note that the formula for the ith x-coordinate is given by $x_i = 1 + 3i/5$.

Using the lefthand rule, the height of the ith rectangle (i.e., the height of the rectangle whose base is given by the interval $[x_{i-1}, x_i]$) is given by $f(x_{i-1})$. Thus the area of the ith rectangle is given by

$$\begin{aligned} \text{Area} &= \text{height} \times \text{width} && (15.5.1)\\ &= f(x_{i-1}) \times 3/5 && (15.5.2)\\ &= \left(9 - (x_{i-1})^2\right) \times 3/5. && (15.5.3) \end{aligned}$$

The Riemann sum is obtained by adding the area of all these rectangles together. Thus, using the lefthand rule, the Riemann sum is given by

$$\begin{aligned}\text{Riemann sum} &= \text{Sum of the areas of all the rectangles} && (15.5.4)\\ &= \sum_{i=1}^{n} \text{Area of } i\text{th rectangle} && (15.5.5)\\ &= \sum_{i=1}^{5} f(x_{i-1}) \times 3/5 && (15.5.6)\end{aligned}$$

This last expression is a possible answer for the Riemann sum (in Sigma notation). If you wanted to actually write out the sum, you would find

$$\begin{aligned}\sum_{i=1}^{5} f(x_{i-1}) \times 3/5 &= f(x_0) \times 3/5 + f(x_1) \times 3/5 + f(x_2) \times 3/5 + f(x_3) \times 3/5 + f(x_4) \times 3/5 && (15.5.7)\\ &= (f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)) \times 3/5. && (15.5.8)\end{aligned}$$

Plugging in values of the x_i, we obtain another expression for the Riemann sum:

$$(f(1) + f(8/5) + f(11/5) + f(14/5) + f(17/5)) \times 3/5. \qquad (15.5.9)$$

Note that there are indeed five summands, corresponding to the five rectangles we used. Now, remembering what f is, we obtain yet another expression for the Riemann sum. It is the most explicit and computable:

$$\left(9 - (1)^2 + 9 - (8/5)^2 + 9 - (11/5)^2 + 9 - (14/5)^2 + 9 - (17/5)^2\right) \times 3/5. \qquad (15.5.10)$$

So the expressions (15.5.6), (15.5.8), (15.5.9), and (15.5.10) are all acceptable answers for setting up the Riemann sum using the lefthand rule. Which of these looks most concise to you?

For the righthand rule, the height of the ith rectangle (i.e., the height of the rectangle with base $[x_{i-1}, x_i]$) is given by $f(x_i)$. So the area of the ith rectangle is given by

$$f(x_i) \times 3/5.$$

Adding the areas for all the rectangles, we find that the Riemann sum for the righthand rule is given by

$$\sum_{i=1}^{5} f(x_i) \times 3/5 = (f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) \times 3/5.$$

Plugging in the values for the x_i, we find that this equals

$$(f(8/5) + f(11/5) + f(14/5) + f(17/5) + f(4)) \times 3/5.$$

I won't write out what $f(8/5)$ is, nor the other terms; you can do that.

For the midpoint rule, the height of the ith rectangle is given by f of the midpoint of the ith interval. The ith interval is given by $[x_{i-1}, x_i]$, so its midpoint is given by $(x_i + x_{i-1})/2$. Hence the area of the ith rectangle using the midpoint rule is

$$f(\frac{x_i + x_{i-1}}{2}) \times 3/5.$$

And the Riemann sum, in Σ notation, is given by

$$\sum_{i=1}^{5} f(\frac{x_i + x_{i-1}}{2}) \times 3/5.$$