Lecture 12

Review and practice problems for Exam One

Exercise 12.0.1. Consider the shape given by those points (x, y) satisfying

$$\cos(y)x + y\sin(x) = 0.$$

Find the slope of the line tangent to this shape at a point $(\pi/4, -\pi/4)$.

A possible solution. Take the derivative of both sides of the original equation to find:

$$-\sin(y)y'x + \cos(y) + y'\sin(x) + y\cos(x) = 0. \tag{12.0.1}$$

Isolate the y' to find:

$$y' = \frac{-y\cos(x) - \cos(y))}{-x\sin(y) + \sin(x)}.$$

Now we plug in $x = \pi/4$ and $y = -\pi/4$ to find:

$$\begin{aligned}\frac{\frac{\pi}{4}\cos(\frac{\pi}{4}) - \cos(-\frac{\pi}{4})}{-\frac{\pi}{4}\sin(-\frac{\pi}{4}) + \sin(\frac{\pi}{4})} &= \frac{\frac{\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{-\frac{\pi}{4}\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} \\ &= \frac{(\pi - 4)\sqrt{2}}{(\pi + 4)\sqrt{2}}.\end{aligned}$$

Comments: One tricky part is noting that $(\cos(y))' = -\sin(y)y'$. Note also that we had to use the product rule to compute the derivative of $\cos(y)x$ and of $y\sin(x)$. $\square$

Exercise 12.0.2. You are designing a rectangular dance floor. As it turns out, the material for the dance floor itself costs 100 dollars per square foot; you also want to line *one* edge of the dance floor with material A which costs 50 dollars per foot, while lining the other three edges with material B which costs 20 dollars per foot. You want your dance floor to be exactly 1000 square feet.

(a) How many meters should the **Material A** wall[1] be to minimize the total cost of the dance floor?

(b) What is the minimal cost?

A possible solution. What is the problem asking us to do? To minimize **cost**.

So, what is cost? Let's set some variables. Let's say that A is the length of the edge made of material A, measured in feet. And let's say that W is the length of an edge perpendicular to the edge made of material A, also measured in feet.

Note that the wall made of material A then costs $50A$ dollars.

Likewise, the other three edges cost $20W + 20W + 20A$ dollars total.

And, the dance floor itself (without the edges) costs 1000×100 dollars.

So the total cost for the dance floor is

$$70A + 40W + 100000.$$

And the problem asks us to minimize the cost.

Now, we don't know how to minimize the cost of a function that involves *two* input variables.[2] So how can we make the cost function depend only on one variable?

The problem gives us a relation between A and W: Because the area must be exactly 1000 square feet, we know

$$AW = 1000. \tag{12.0.2}$$

So by using (12.0.2), we can use substitution to get rid of one of the two input variables in our cost function. For example, if you dislike A, you can set

$$A = \frac{1000}{W}$$

and substitute into the cost function to make cost only a function of W:

$$\text{Cost} = 70\frac{1000}{W} + 40W + 100000.$$

[1] Fixed a typo.

[2] That's for Calculus III.

Now, how do we try to find minima of a function depending on a variable? We begin by finding local minima.

How do we find local minima? By finding critical points.[3] Then, we test the second derivative at these critical points to see if the critical points are local minima or maxima.

So let's find the critical points. This means finding out where the second derivative equals zero. Well,

$$\frac{d\text{Cost}}{dW} = -140000W^{-2} + 40.$$

So for the derivative to equal zero means

$$W^2 = \frac{40}{140000}.$$

Thus

$$W = \sqrt{\frac{40}{140000}} = 2\sqrt{\frac{1}{3500}} = \frac{2}{10}\sqrt{\frac{1}{35}} = \frac{1}{5}\sqrt{\frac{1}{35}} = \frac{1}{175}\sqrt{35}.$$

To check that this is a local minimum, we should compute the second derivative of cost at this point. The secon dderivative is

$$280000W^{-3}$$

which is positive when W is positive. So indeed $W = \frac{1}{175}\sqrt{35}$ is a local minimum.

Part (a) asks for the length of the material A wall. Well,

$$A = \frac{1000}{W} = 1000 \times 175/\sqrt{35} = 5000 \times \sqrt{35}.$$

So the answer is $5000\sqrt{35}$ feet. (This is a very long and narrow dance floor!)

Part (b) asks for the cost. Well, cost is given by

$$70A + 40W + 100000 = 70 \times 5000 \times \sqrt{35} + 40 \times \frac{1}{175}\sqrt{35} + 100000.$$

You can simplify this expression if you like, or not.

Comments: Look back at the questions that were asked in this possible solution. These are the kinds of questions that will help you during an exam. □

Exercise 12.0.3. A cylindrically shaped rod is expanding in the oven. Suppose:

[3] By definition, a critical point is where the derivative equals zero.

1. The length of the rod is 300 millimeters,
2. The radius of the rod is 10 millimeters, and
3. The length is not changing, but the volume is growing at 3 millimeters-cubed per second.

In millimeters per second, how quickly is the radius changing?

Present your answer in millimeters per second and exactly—this means π will show up somewhere in your answer.

(Remember that the volume of a cylinder is given by

$$V = \pi r^2 l$$

where r is the radius of the cylinder and l is the length.)

A possible solution. Take the derivative of volume to find:

$$V' = 2\pi r r' l + \pi r^2 l'.$$

(Note we used the product rule.) The problem tells us that $l = 300, r = 10, l' = 0, V' = 3$. Thus

$$3 = 2\pi \times 10 \times r' \times 300 + 0.$$

So we find that

$$r' = \frac{3}{2\pi \times 10 \times 300} = \frac{1}{2000\pi}.$$

□

Exercise 12.0.4. The CDC is using a function $f(t)$ to model the number of hospitalizations due to a new disease, where t is in years, and $f(t)$ is the number of patients in the hospital at time t. For example, $f(3)$ measures the number of patients in the hospital due to the disease 3 years from now.

While the CDC does not tell us $f(t)$, they have recently told your team of researchers that the *derivative* of $f(t)$ is given by the following function:

$$\frac{9 - 9t^2}{e^t + 1}.$$

According to this, when would you expect the number of patients hospitalized due to this disease to be the greatest?

A possible solution. The given fraction is the *derivative* of f. So to find the local extrema of f, we should find where this fraction equals zero. So we have to solve for those values of t where

$$0 = \frac{9 - 9t^2}{e^t + 1}.$$

Well, for "blah / something" to equal zero, blah must equal zero. So we are solving for

$$0 = 9 - 9t^2.$$

This has two solutions: $t = 1$ and $t = -1$. Now, let's compute the second derivative of f to see which of these is a local max.

$$\begin{aligned} \left(\frac{9 - 9t^2}{e^t + 1}\right)' &= \frac{(-18t)(e^t + 1) - (9 - 9t^2)(e^t)}{(e^t + 1)^2} \\ &= \frac{(9 - 18t - 9t^2)e^t - 18t}{(e^t + 1)^2}. \end{aligned}$$

Plugging in $t = 1$ (one of our critical points), we find that this second derivative is

$$\frac{(9 - 18 - 9)e - 18}{(1 + 1)^2} = \frac{-18e - 18}{4}$$

which is a negative number. This means that $t = 1$ is a local maximum (because the function is concave down there).

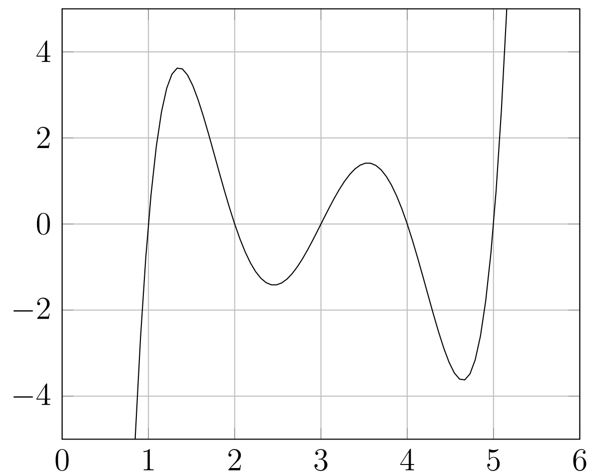
Let's plug in $t = -1$ to see if this is a local maximum:

$$\frac{(9 + 18 - 9)e^{-1} + 18}{(e^{-1} + 1)^2} = \frac{18e^{-1} + 18}{(1 + 1/e)^2}.$$

This is a positive number, so $t = -1$ is a local minimum.

In short, the local maximum is at $t = 1$, so we expect the number of patients hospitalized to be greatest at $t = 1$ years from now. □

Exercise 12.0.5.



Above is the graph of a function $g(x)$.

(a) In the above graph, **bold** (or shade over) the portions of the graph where $g(x)$ has *negative* second derivative.

(b) In the above graph, draw a **clearly visible dot** at each inflection point of g.

Exercise 12.0.6. 1. Hiro wants to find a local minimum for the function $f(x) = e^x \sin(x)$. Which of the following equations should Hiro solve to start finding a local minimum?

(a) $e^x \sin(x) = 0$

(b) $e^x(2\sin(x)) = 0$

(c) $e^x(\sin(x) + \cos(x)) = 0$

(d) $e^x \sin(x) \geq e^y \sin(y)$

(e) $e^x = 0$

(f) $e^{x\sin(x)} = 0$

2. Hiro is trying to find local maxima of a function $g(x)$. He has found a point a where the derivative is zero. In which of the following scenarios does Hiro know that a is a local maximum?

(a) $g'(a) = 0$.

(b) $g''(a) = 0$.

(c) $g'(a) > 0$.

(d) $g'(a) < 0$.

(e) $g''(a) < 0$.

(f) $g''(a) > 0$.

3. Hiro is studying a function $h(x)$, and he has found a point x at which $h'(x) = 0$ and $h''(x) = 0$. Which of the following must be true?

(a) Hiro has found a local maximum.

(b) $h(x) = 0$.

(c) $h'''(x) = 0$.

(d) Hiro has found an inflection point.

(e) Hiro has found a local minimum.

(f) Hiro cannot tell with the information given whether x is a local maximum.

Exercise 12.0.7. Consider the shape given by those points (x, y) satisfying

$$e^x + ye^x = 0.$$

Find the slope of the line tangent at a point (x, y) on this shape.

Exercise 12.0.8. You are designing a rectangular pen. The area contained in this pen must be exactly 80 square meters. One wall of this pen must be made of Material A, and Material A costs 300 dollars per meter of wall. The other three walls must be made of Material B, and Material B costs 200 dollars per meter of wall.

(Note that you don't need to worry about heights of wall in this problem; pricing is per meter of length along the perimeter of the pen.)

(a) How many meters should the Material B wall be to minimize the total cost of the walls of the pen?

(b) What is the minimal cost?

Exercise 12.0.9. Consider the shape given by those points (x, y) satisfying

$$y^2 + y^3 = 3x - 9.$$

Find the slope of the line tangent at a point (x, y) on this shape.

Exercise 12.0.10. Consider the shape given by those points (x, y) satisfying

$$9(x - 3)^2 + (y - 2)^3 = 1.$$

Find the slope of the line tangent at a point (x, y) on this shape.

Exercise 12.0.11. A zookeeper is designing a rectangular pen for a lion. The area contained in this pen must be exactly 40 square meters. One wall of this pen must be made of sturdy glass, and sturdy glass walls cost 300 dollars per meter of wall.

The other three walls must be made of brick, and brick walls cost 250 dollars per meter of wall.

(Note that you don't need to worry about heights of wall in this problem; pricing is per meter of length along the perimeter of the pen.)

(a) How many meters should the glass wall be to minimize the total cost of the walls of the pen?

(b) What is the minimal cost?