Lecture 11

Implicit differentiation

Remark 11.0.1. The images in this lecture were generated using Desmos.

We now know how to take derivatives of $f(x)$ if we are given a formula stating "$f(x)$ is this." For example, if $f(x) = xe^x$, you know how to take its derivative. This situation, geometrically, allows us to find (slopes of) tangent lines to *graphs*. Algebraically, we know we can reduce the problem of taking derivatives to very basic rules (product and chain rule, along with basic knowledge about sin, e^x, cos, and polynomials).

Today we're going to use the power of the chain rule to expand our minds both algebraically and geometrically. The computational technique we're about to learn is called *implicit differentiation*. There are two huge uses of implicit differentiation:

(a) (Geometric.) To find (slopes of) tangent lines to shapes that are *not* graphs.

(b) (Algebraic.) To find rates of change of functions for which we may not know a formula, but for which we do know a constraint/relation.

11.1 A warm-up: Chain rule with abstract functions

By now, you're familiar with the chain rule. So you know that

$$\frac{d}{dx}(\sin(x))^2 = 2\sin(x)\cos(x).$$

But now suppose that you had a mystery function called g instead of sin:

$$\frac{d}{dx}(g(x))^2 = ?$$

Well, because you're internalized the chain rule, you know how to take the above derivative: Take the derivative of the outside function, plug in the inside function, and multiply by the inside function's derivative. The end result is:

$$\frac{d}{dx}(g(x))^2 = 2g(x)g'(x).$$

Exercise 11.1.1. Take the derivative of each function below.

(a) $e^{g(x)}$.

(b) $\sin(h(x))$.

(c) $h(x)g(x)$.

(d) $\ln(f(x))$.

(e) $(r(x))^2$.

(f) $g(x)^3$.

Solutions. (a) $g'(x)e^{g(x)}$.

(b) $\cos(h(x))h'(x)$.

(c) $h'(x)g(x) + h(x)g'(x)$.

(d) $\frac{f'(x)}{f(x)}$.

(e) $2r(x)r'(x)$.

(f) $3g(x)^2g'(x)$.

□

11.2 An algebraically useful example: The power rule for fractions

Exercise 11.2.1. Here is a fact: The power rule is true for *any* exponent, whether the exponent is a whole number or not. For example,

$$(x^{1/5})' = \frac{1}{5}x^{-4/5}.$$

If you know the usual power rule for whole numbers, is there a way you can verify (or prove) the above equality? In other words, why is the above equality true?

A possible solution using implicit differentiation! Let's think about the function $f(x) = x^{1/5}$. This is the "fifth root" function. We've already used in various problems that the power rule

$$(x^a)' = ax^{a-1}$$

works for whole numbers (positive or negative), but how about for fractions? Here's a very clever trick:

If $f(x)$ is the fifth root of x, then we know

$$f(x)^5 = x.$$

This is an equality of functions. On the lefthand side is a new function, called "do f, then raise to the 5th power" while the righthand side is a boring function called "if you input x, output x." Because the two functions are equal, so is their derivative:

$$5f(x)^4 f'(x) = 1. \tag{11.2.1}$$

Let us now divide both sides by $5f(x)^4$, to find

$$f'(x) = \frac{1}{5f(x)^4}. \tag{11.2.2}$$

□

Exercise 11.2.2. Remember that $f(x) = x^{1/5}$. Using (11.2.2), show why the power rule for $x^{1/5}$ is true.

Possible solution. Reminding ourselves that $f(x) = x^{1/5}$, we find

$$f(x)^4 = (x^{1/5})^4 = x^{4/5}. \tag{11.2.3}$$

Combining (11.2.2) and (11.2.3), we find:

$$f'(x) = \frac{1}{5x^{4/5}}$$

or, just re-arranging and remembering that negative exponents represent division,

$$f'(x) = \frac{1}{5}x^{-4/5}.$$

That is,

$$(x^{1/5})' = \frac{1}{5}x^{-4/5}.$$

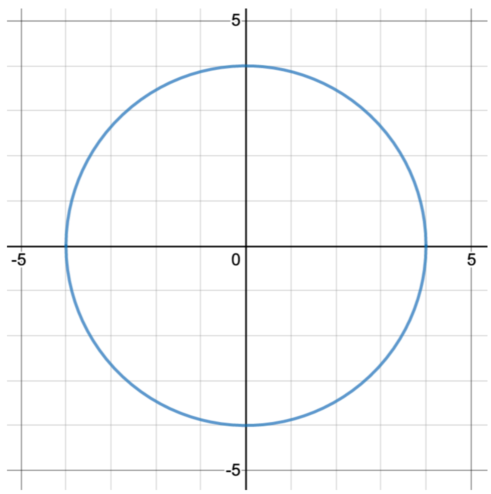
□

So we just verified the power law works when the exponent is 1/5. How cool is that? In fact, this same technique works for *any* fraction of the form 1/n where n is a whole number. Give it a shot for practice!

The key step in the above was line (11.2.1) – more specifically, the process to get to (11.2.1) given the equality before it. The idea is that if you know some constraints/relations that a function f has to satisfy, then you gain information about the derivative of that function.

11.3 A geometric example: The circle

Consider the circle of radius 4:



As you know, the circle is *not* the graph of any function. For example, the circle fails the vertical line test.[1]

But is there a way we might be able to figure out the slope of the line tangent to the circle, at say, the point $(1, \sqrt{15})$?

As you know, the circle of radius 4 is defined as the set of points (x, y) satisfying the relation

$$x^2 + y^2 = 16.$$

Well, let's pretend for a moment that this number y is a function of x. If you like, we can make this explicit by writing $y = f(x)$, and substituting:

$$x^2 + (f(x))^2 = 16. \tag{11.3.1}$$

[1] I expect you to know about the vertical line test from precalculus.

Exercise 11.3.1. Using (11.3.1),

(a) find an expression of $f'(x)$ in terms of $f(x)$ and x.

(b) What is the slope of the tangent line to the circle of radius 4 at the point $(1, \sqrt{15})$?

Possible solution. Here is some work you might do on an exam, starting with (11.3.1):

$$\begin{aligned} x^2 + (f(x))^2 &= 16 \\ \implies (x^2 + (f(x))^2)' &= (16)' \\ 2x + 2f(x)f'(x) &= 0 \\ 2f(x)f'(x) &= -2x \\ f(x)f'(x) &= -x \\ f'(x) &= \frac{-x}{f(x)} \end{aligned} \tag{11.3.2}$$

This completes (a): We found an expression for $f'(x)$ where we can calculate $f'(x)$ knowing only x and $f(x)$.

(b) Remember that "$f(x)$" was our stand-in for the y-coordinate. (We were pretending that y is a function of x.) So at the point $(1, \sqrt{15})$, we have that $x = 1$ and $f(x) = f(1) = \sqrt{15}$. Hence

$$f'(x) = \frac{-1}{\sqrt{15}}.$$

□

11.4 Using y instead of $f(x)$

Using the substitution $y = f(x)$ allows you to "peek under the hood" of how we are really thinking about the problem. We pretend y is a function, apply the rules of derivatives we know, and we manage to find the slope to a tangent line at the point (x, y).

But some calculus textbooks, and many mathematicians, are so lazy they would rather not make the substitution. So here is the way one might write out the work from before. (Compare to (11.3.2).)

$$\begin{aligned} x^2 + y^2 &= 16 \\ \implies (x^2 + y^2)' &= (16)' \\ 2x + 2yy' &= 0 \\ 2yy' &= -2x \\ yy' &= -x \\ y' &= \frac{-x}{y}. \end{aligned}$$

This notation can be very confusing: y was just a coordinate, so why are we allowed to take y'? Why does it have a derivative? Again, the under-the-hood mechanism is that we are pretending y is a function to justify this notation.

There is nothing inherently better about the y notation to the $f(x)$ notation; but you will see both notations in the world, so I wanted to expose you to both.

11.5 Summary

The common trait that the above algebraic and geometric examples have is that sometimes, an expression of the form $f(x) = ...$ is either unavailable or not as useful for taking derivatives. Regardless, a constraint, or an equation involving f, is often enough to know f' when you know particular values of x and $f(x)$.

Exercise 11.5.1. Suppose that you have a function f, and you know that every point on the graph of f satisfies the following equality:

$$x^2 + f(x)^2 = 25.$$

(a) Hiro tells you that he thinks the point $(3, 4)$ is a point on the graph of f. Is Hiro's claim consistent with the given information?

(b) What is the slope of the tangent line to the graph of f at that point?

A possible solution. (a) For this part, let's just plug in $x = 3$ and $f(3) = 4$; if things are consistent, then these values should satisfy the given equality. Indeed:

$$3^2 + f(3)^2 = 9 + 4^2 = 9 + 16 = 25.$$

(b) f is a mysterious function indeed! And $x^2 + f(x)^2$ is another mysterious function (built using f), but we are given an amazing fact: That this mysterious new function is *constant* (with value 25).

So if we take its derivative, we find that

$$\begin{aligned} & (x^2 + f(x)^2)' = 25' \\ \implies & 2x + 2f(x)f'(x) = 0. \end{aligned} \tag{11.5.1}$$

This differentiation is very similar to the way we took the derivative of $r(t)^2$ in the related rates lecture. To go through it again: $f(x)^2$ is a function with input x, and it can be expressed as the composition of an inside function called $f(x)$ with an outside function called squaring. So we used the chain rule to take the derivative of $f(x)^2$ in the above.

The problem asks for the slope of the tangent line to a graph – in other words, it asks for f' at a particular point. Well, the above equation can be finagled to isolate $f'(x)$ on one side of the equality symbol:

$$\begin{aligned} 2x + 2f(x)f'(x) &= 0 \\ 2f(x)f'(x) &= -2x \\ f'(x) &= \frac{-2x}{2f(x)} \\ f'(x) &= \frac{-x}{f(x)}. \end{aligned} \tag{11.5.2}$$

And, if we are at the point $(3, 4)$, we plug in $x = 3$ and $f(x) = f(3) = 4$, to find

$$f'(x) = \frac{-3}{4}.$$

□

Remark 11.5.2. In the above example, we never needed to find a formula for $f(x)$ to compute $f'(x)$!

If somebody told you "I don't know a formula for $f(x)$, but I do know that it satisfies some relation" you may not have known how to find $f'(x)$ before today.

Here is the **take-away**: Even without knowing explicitly what $f(x)$ is, if we know some other relation that $f(x)$ satisfies, we can compute its derivative!

Put another way, we can find derivatives for functions where we are not given a formula of the form "$f(x) = \ldots$." As a rule of thumb, even if we do not have a formula for $f(x)$ in terms of x, so long as we know a *value* of $f(x)$ at some number x, we can often find the value of $f'(x)$ there.

Example 11.5.3. In Exercise 11.5.1, we never figured out a formula for $f(x)$ itself. Regardless, by knowing $f(3)$, we were able to compute $f'(3)$. The key step was (11.5.1) – which involved taking the derivative of the one thing we knew about $f(x)$. The rest was algebra we performed to isolate $f'(x)$, allowing us to write down an expression for $f'(x)$ in terms of $f(x)$ and x in (11.5.2).

This is rather magical. This is the hidden power of the chain rule. How cool is that?

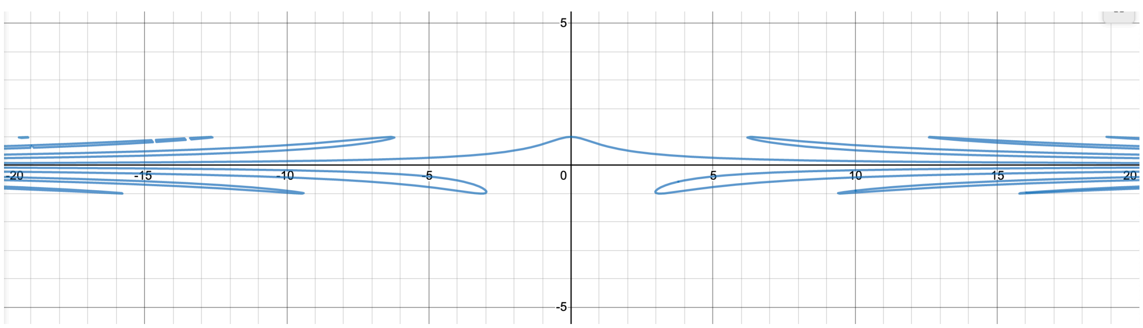
11.6 Many shapes aren't graphs

Example 11.6.1. In science, it happens all the time that we look for solutions to equations like the following:

$$y - \cos(xy) = 0.$$

The key point here is that the appearances of *y cannot be separated* from the functions and variables; so it is either difficult, or impossible, to put the above equations into the form $y =$ (something involving only x). So we'll rarely find that the set of all points satisfying the above equation is a graph of something.

Can you plot all the points (x, y) on the plane so that the above equation is satisfied? What does the shape look like? This turns out to be very hard; in case you're curious, here's a bit of the solution set. It looks even more interesting as you zoom out from what I've drawn here.

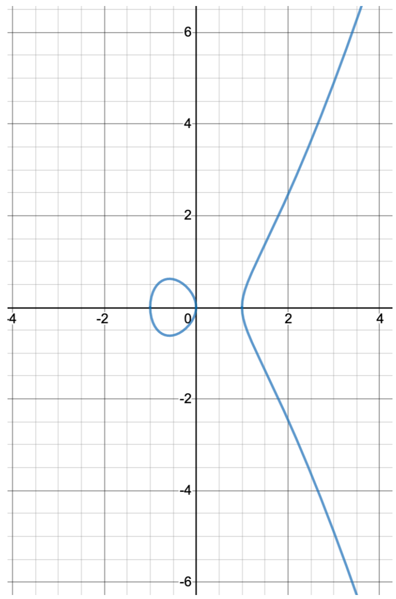


(This solution set is definitely not the graph of some function; it fails the vertical line test.)

Regardless, let's say you can find some point that solves the above equation. Can you at least find the slope (of the tangent line) at that point? Then, even if you can't visualize the above shape, you can still see very interesting information!

Example 11.6.2. Another example is below; it's something called an *elliptic curve*, and in this case, we're plotting all those points (x, y) satisfying

$$y^2 = x^3 - x.$$



11.7 Another worked out example

Implicit differentiation *pretends* that y is a function of x, and then takes the derivative. Let me say what I mean.

Assume you have a function f, and that you know the function satisfies the following equation for all x:

$$f(x) - \sin(xf(x)) = 12.$$

Well, this says that there's a function on the lefthand side, and a (constant) function on the righthand side, and they're equal; so their derivatives must be equal! Let's take the derivatives of both sides.

$$f'(x) - \cos(xf(x))(f(x) - xf'(x)) = 0.$$

(On the left, I've used the chain rule.) We can rearrange terms to find:

$$f'(x) = \frac{f(x)\cos(xf(x))}{1 - x}.$$

In other words, we have found the derivative of f *in terms of* x *and* $f(x)$—if we know x and we know $f(x)$, we know the slope of f there.

Notation 11.7.1 (y and y' in implicit differentiation)**.** Notationally, here is how implicit differentiation is carried out. So suppose instead the you are curious about the shape formed by the equation

$$y - \sin(xy) = 12.$$

We are going to **pretend y is a function of x**, and we will take the derivatives of both sides:

$$y' - \cos(xy)(y - xy') = 0.$$

Then, we solve for y':

$$y' = \frac{y\cos(xy)}{1-x}.$$

Note that this answer is *identical* to the above answer, with $f(x)$ replaced by y. Here is how to interpret this equation: On the lefthand side is the slope of my shape, and on the righthand side is an expression for that slope in terms of x and y. Put another way, *if I know where I am, I know the slope of my shape there.* Here, "where I am" is given by the value of x and y I plug into the righthand side—it's given by the point (x, y) on the plane.

Exercise 11.7.2. Consider the ellipse given by the equation

$$3\,(x-3)^2 + (y-1)^2 = 2.$$

Find the slope of the tangent line to the ellipse at a point (x, y) on the ellipse.

Possible solution. We take the derivative of both sides of the above equation, pretending that y is a function of x. Then we get

$$\left(3\,(x-3)^2 + (y-1)^2\right)' = (2)'.$$

The lefthand side becomes

$$\left(3\,(x-3)^2 + (y-1)^2\right)' = (3\,(x-3)^2)' + ((y-1)^2)' = 6(x-3) + 2(y-1)y'.$$

Thus

$$6(x-3) + 2(y-1)y' = 0.$$

Now we rearrange the equation so that y' is alone:

$$y' = \frac{-3(x-3)}{y-1}.$$

This gives the answer. □

Remark 11.7.3. For example, you can check that the point $(3, 1+\sqrt{2})$ is on this ellipse. Then the slope of the tangent line there is given by

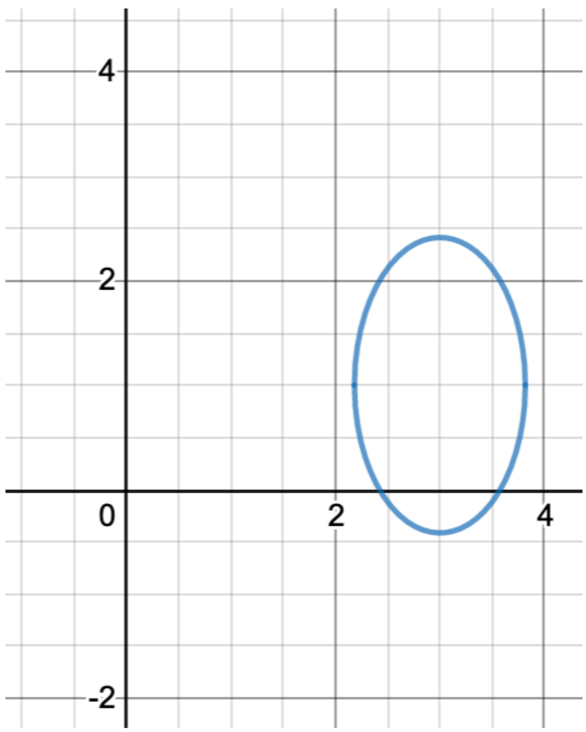
$$\frac{-3(3-3)}{1+\sqrt{2}-1} = 0.$$

You can also check that the point $(3+\sqrt{1/3}, 2)$ is on this ellipse. The slope of the tangent line there is given by

$$\frac{-3(3+\sqrt{1/3}-3)}{2-1} = -3\sqrt{1/3}.$$

Note also that y' approaches infinity as y approaches 1. Indeed, these are points at which the tangent line becomes vertical.

Here is a picture of the ellipse in case you want to study our results further:



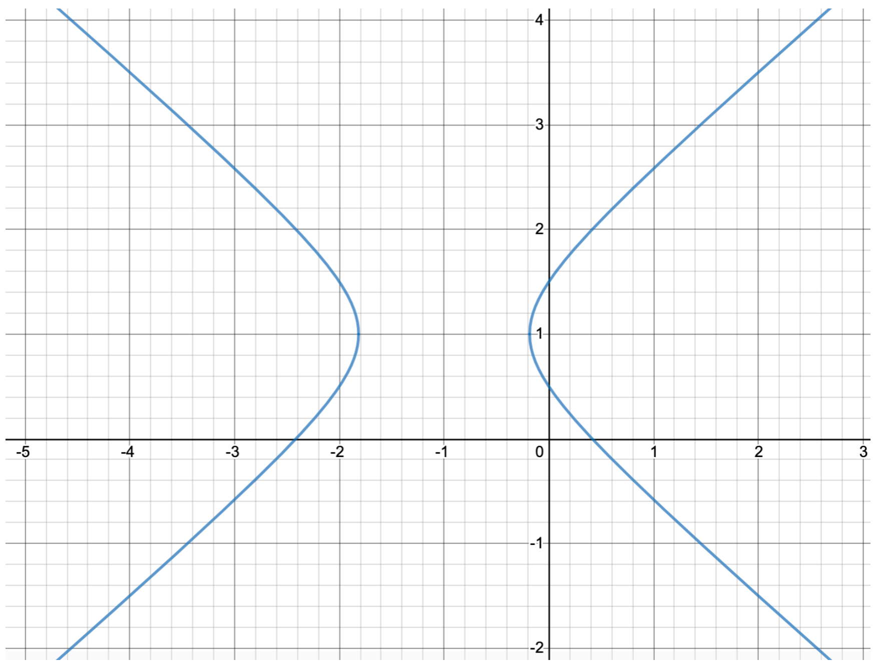
Exercise 11.7.4. Here is an equation for a hyperbola:

$$3\left(x+1\right)^2 - 4\left(y-1\right)^2 \;=\; 2.$$

(i) Using implicit differentiation, find a formula for the slope of the tangent line to this hyperbola at a point (x, y).

(ii) How does this slope behave as x approaches ∞? (Is there a single behavior?) Be warned: This is a fun problem and will take a little trickery!

(iii) Below is an image of the hyperbola. What does your answer to part (ii) have to do with this picture?



Exercise 11.7.5. The equation below defines a beautiful shape (see Example 11.6.1):

$$y - \cos(xy) = 0.$$

Find a formula for the slope of the tangent line to this shape at the point (x, y).

Possible solution.

$$\begin{aligned} y - \cos(xy) &= 0 \\ \implies (y - \cos(xy))' &= (0)' \\ y' + \sin(xy) \cdot (xy)' &= 0 \\ y' + \sin(xy) \cdot (y + xy') &= 0 \\ y' + y\sin(xy) + xy'\sin(xy) &= 0 \\ y'(1 + x\sin(xy)) + y\sin(xy) &= 0 \\ y'(1 + x\sin(xy)) &= -y\sin(xy) \\ y' &= \frac{-y\sin(xy)}{1 + x\sin(xy)} \end{aligned}$$

□

11.8 Bonus: Comparing two methods

A clever way you might think of to find the slope of the tangent line. One observation you might make is that the circle is not a graph, but it's *made up of graphs*. Here's what I mean. The circle of radius 4 is defined as the set of points (x, y) satisfying the relation

$$x^2 + y^2 = 16.$$

In a past life, you may have done some algebra to write y as a function of x:

$$\begin{aligned} x^2 + y^2 &= 16 \\ y^2 &= 16 - x^2 \\ y &= \pm\sqrt{16 - x^2} \end{aligned}$$

where y could be a positive or a negative number, so we write "±" to remind ourselves that y could be the positive or the negative square root of $16 - x^2$.

The meaning of the above algebra is that *if we know the x coordinate, then we know that the y coordinate is (plus or minus) the square root of* $16 - x^2$. In other words, y looks like a function of x:

$$y = f(x), \qquad \text{where } f(x) = \sqrt{16 - x^2} \text{ or } f(x) = -\sqrt{16 - x^2}.$$

Then the slope of the tangent line can be found by taking the derivative of this functions. Let's take the positive function for concreteness – that is, let's assume

$f(x) = \sqrt{16 - x^2}$, so that we are only looking for tangent lines where y is positive. We compute, using the chain rule and the power rule:

$$\begin{aligned} f'(x) &= (\sqrt{16 - x^2})' \\ &= ((16 - x^2)^{1/2})' \\ &= \frac{1}{2}(16 - x^2)^{-1/2}) \cdot (16 - x^2)' \\ &= \frac{1}{2}(16 - x^2)^{-1/2}) \cdot (-2x) \\ &= -x(16 - x^2)^{-1/2}) \\ &= \frac{-x}{\sqrt{16 - x^2}}. \end{aligned}$$

So this tells you the tangent line to the circle at the point with positive y coordinate associated to x, and indeed this agrees with our answer from before (see (11.3.2)) because $y = \sqrt{16 - x^2}$. □

So, some problems can be answered without implicit differentiation – but such methods are often more tedious!

11.9 For next time

You should be able to do the following problem (and problems similar to it):

Consider the collection of all points (x, y) on the xy-plane satisfying the equation

$$(x - 3)^2 + (y - 3)^2 = 9.$$

(a) Write down a formula for the slope of the tangent line to this shape at a point (x, y) on this shape.

(b) Find the slope of the tangent line to this shape at the point $(0, 6)$.

(c) Find the slope of the tangent line to this shape at the point $(1, 3 + \sqrt{5})$.