Lecture 10

Related Rates

Sometimes, we want to know the rate at which something is changing, but we are initially on giving information about the rate of change of a *related* quantity. Calculus allows us to compute the rate we want to compute based on information about the related quantity.

Problems that help us convert rate-of-change information about some quantities into rate-of-change information about another quantity are called *related rates* problems.

10.1 A warm-up: Chain rule with abstract functions

By now, you're familiar with the chain rule. So you know that

$$\frac{d}{dx}(\sin(x))^2 = 2\sin(x)\cos(x).$$

But now suppose that you had a mystery function called g instead of sin:

$$\frac{d}{dx}(g(x))^2 = ?$$

Well, because you're internalized the chain rule, you know how to take the above derivative: Take the derivative of the outside function, plug in the inside function, and multiply by the inside function's derivative. The end result is:

$$\frac{d}{dx}(g(x))^2 = 2g(x)g'(x).$$

Exercise 10.1.1. The functions below are functions containing functions to which we give only names. Take the derivative of each function below.

(a) $e^{g(x)}$.

(b) $\sin(h(x))$.

(c) $h(x)g(x)$.

(d) $\ln(f(x))$.

(e) $(r(x))^2$.

(f) $g(x)^3$.

Solutions. (a) $g'(x)e^{g(x)}$.

(b) $\cos(h(x))h'(x)$.

(c) $h'(x)g(x) + h(x)g'(x)$.

(d) $\frac{f'(x)}{f(x)}$.

(e) $2r(x)r'(x)$.

(f) $3g(x)^2g'(x)$.

□

10.2 Examples of related rates problems

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Exercise 10.2.1. The area of a crop circle is expanding at a rate of 3 meters squared per minute (i.e., $3\text{m}^2/\text{min}$). If you know what the radius of the crop circle is at a certain time, can you tell me how quickly the radius of this crop circle is increasing at that time?

Exercise 10.2.2. The area of a different crop circle at time t is given by

$$A(t) = e^{3t},$$

where t is in minutes and the area is in meters squared. At time t, how quickly is the radius of this crop circle increasing?

Possible solutions. In tackling each of these problems, we have to think about how the area depends on the radius. Of course, for circles, we know that area is equal to π times the radius squared. That is,

$$A = \pi r^2.$$

Now, the area is changing with time, so the radius is changing with time, too. We can write

$$A(t) = \pi r(t)^2.$$

(Both area and radius are now expressed as functions of time.)

So let's take the derivative of both sides, with respect to t:

$$A'(t) = \pi 2r(t) \cdot r'(t).$$

(We are using the chain rule on the righthand side!) Dividing both sides by $2\pi r(t)$, we find:

$$r'(t) = \frac{A'(t)}{2\pi r(t)}.$$

So, for the first exercise, when the circle has radius R, we know that the radius is changing as

$$r'(t) = \frac{A'(t)}{2\pi r(t)} = \frac{3}{2\pi R}.$$

For the second exercise, we see that

$$A'(t) = (e^{3t})' = 3e^{3t}.$$

Moreover, we can find $r(t)$ in terms of $A(t)$:

$$r(t) = \sqrt{A(t)/\pi} = \sqrt{e^{3t}/\pi} = \frac{e^{3t/2}}{\sqrt{\pi}}.$$

So

$$\begin{aligned} r'(t) &= \frac{A'(t)}{2\pi r(t)} && (10.2.1) \\ &= \frac{3e^{3t}}{\frac{e^{3t/2}}{\sqrt{\pi}}} && (10.2.2) \\ &= \frac{3e^{3t-(3t)/2}}{\frac{1}{\sqrt{\pi}}} && (10.2.3) \\ &= 3\sqrt{\pi}e^{3t/2}. && (10.2.4) \end{aligned}$$

$\square$

10.2.1 r versus $r(t)$

Here is a common question: What is the difference between the notation r, and the notation $r(t)$?

First, you may have noticed that I sometimes call functions f, and I sometimes call them $f(x)$. Here, f is the name of a function – like "Bob" – and $f(x)$ represents the *number* that Bob outputs when I plug in an input called x. And $f'(x)$ represents the number outputted by the derivative of f (the derivative of Bob).

Now, the use of writing $A = \pi r^2$ versus $A(t) = \pi r(t)^2$ is that the latter notation reminds us that area and radius may be changing with respect to a variable called t. So A and r aren't just numbers, they are *functions* that yield different results depending on the input t. So, while you may have thought of $A = \pi r^2$ as a static, unmoving formula that describes the area of a circle that is given to you, the beautiful formula $A(t) = \pi r(t)^2$ conjures the possibility that your circle may be shrinking or expanding as t varies, and regardless, the formula holds true at all values of t.

In word problems, writing the "of t" or "of x" – i.e., by writing $A(t)$ or $A(x)$ – we can remind ourselves that A depends on t or on x, and hence we can meaningfully take a derivative of A.

Remark 10.2.3. At the same time, if you feel you can keep track of what is and is not constant, you don't need to worry about the (t) or (x) notation. For example, you could easily do the following work, too:

$$\begin{aligned} A &= \pi r^2 \\ A' &= \pi (r^2)' \\ &= \pi 2 r r' \\ &= 2\pi r r' \end{aligned} \tag{10.2.5}$$

In other words, the two expressions

$$A'(t) = 2\pi r(t) r'(t) \qquad \text{and} \qquad A' = 2\pi r r'$$

are more or less the same thing; the lefthand, first equality is an equality of numbers you obtain by plugging in values of t. Meanwhile, the second equality is an equality of functions.

10.2.2 The "related rates"

I often say that the hardest part of calculus is the algebra. But related rates problems do involve calculus, and in a very important place.

Remember, related rates problems allow you to relate the rate of change of one thing in terms of other quantities we might know.

So the most important calculus step in a related rates problem is finding the way that the desired rate of change is related to other quantities. In the above example, it was the step of taking

$$A(t) = 2\pi r(t)^2 \tag{10.2.6}$$

and taking the derivative of both sides:

$$A'(t) = 2\pi r(t) r'(t). \tag{10.2.7}$$

Let me mentioned that the equation (10.2.6) is an equality of *functions*. (It says that the function on the left is equal to the function on the right.) So if we take the derivatives of both sides, the results are still equal.

The equality (10.2.7) holds the most important piece of information. For example, if we want to know the rate of change of area at a particular time, all we need to know is r and r' at that time.

10.2.3 Another exercise

Exercise 10.2.4. An oil spill has occurred in the Gulf of Mexico – a tanker is leaking oil and a circular patch of oil on the water's surface is increasing in size by the minute.

About 100 minutes into the spill, the circular patch is observed to be 500 meters in radius and the radius appears to be growing at a rate of one meter per second. Assuming that every meter-squared of oil contains half a liter of oil, at what rate is oil being leaked, 100 minutes into the spill? (Give your answer in liters per second.)

Remark 10.2.5. The above exercise is a nice example of some real-life scenarios where knowing how to do related rates can come in handy. It is a lot easier to make one measurement about how quickly radius is changing (go to the edge of an oil spill and measure how far it travels) than to, say, try to take many birds-eye-view photos of an oil spill and try to estimate the rate of change of area. In other words, it happens all the time in real life that we may be able to measure some quantities, but not others – the mathematics here allows us to deduce information anyway. Very powerful!

10.3 Related rates problems involving multiple dependencies

Exercise 10.3.1. Both the radius and length of a cylinder-shaped popsicle are changing over time. Remember that the volume of a cylinder is given by

$$V = \pi r^2 l$$

where r is the radius of the cylinder and l is the length of the cylinder.

(a) To determine the rate of change of V (with respect to time) at time t, which of the following do you need to know? (Select all that apply.)

   (1) The value of $V(t)$.

   (2) The value of $r(t)$.

   (3) The value of $l(t)$.

   (4) The value of $r'(t)$.

   (5) The value of $l'(t)$.

(b) At time $t = 3$ seconds, you are told that the radius is changing at 3 millimeters per second, and that the length is changing at 2 millimeters per second. You are further told that the cylinder is 60 millimeters long with radius 10 millimeters.

   Given this information, how quickly is the volume of the cylindrical popsicle changing, in units of cubic millimeters per second?

Possible solution. To compute the rate of change of volume, we must take its derivative. Noting that both r and l are functions of time t, we can compute $V'(t)$ using the product rule:

$$V'(t) = \pi(2r(t)r'(t)l(t) + r(t)^2 l'(t)).$$

So to compute the lefthand side, we need to know $r(t)$, $r'(t)$, $l(t)$, and $l'(t)$.

The problem tells us that $r'(3) = 3, l'(3) = 2$, and that $l(3) = 60$ and $r(3) = 10$. Plugging in these numbers, we find

$$\begin{aligned} V'(t) &= \pi(2 \cdot 10 \cdot 3 \cdot 60 + 10^2 \cdot 2) \\ &= \pi(3600 + 200) \\ &= 3800\pi. \end{aligned}$$

□

Exercise 10.3.2. The temperature T of an ideal gas depends on the pressure P and volume V of the gas as follows:

$$PV = kT$$

where k is some number. Assume that the air inside a balloon is an ideal gas.

(a) To determine the rate of change of kT (with respect to time) at time t, which of the following do you need to know? (Select all that apply.)

(1) The value of $P(t)$.
(2) The value of $V(t)$.
(3) The value of $T(t)$.
(4) The value of $k(t)$.
(5) The value of $P'(t)$.
(6) The value of $V'(t)$.

(b) You are being told that the balloon is increasing in volume at 2 cubic centimeters per second, and decreasing in pressure at 3 pascals per second. You are further told that the balloon has a volume of 30 cubic centimeters at time $t = 2$ seconds, and that the air in the balloon has pressure exactly 1 pascal at $t = 2$ seconds.

Then, at time $t = 2$, how quickly is the quantity kT changing? Give me your answers in pascal-cubic-centimeters-per-second.

Possible solution. The derivative of kT is computed, using the product rule, as

$$P'(t)V(t) + P(t)V'(t).$$

So to compute this quantity, we need to know $P'(t), V(t), P(t)$, and $V'(t)$.

The problem tells us that at $t = 2$, we have $V'(t) = 2, P'(t) = -3, V(t) = 30, P(t) = 1$, so

$$P'(t)V(t) + P(t)V'(t) = -3 \cdot 30 + 1 \cdot 2 = -27 + 2 = -25.$$

The answer is -25 pascal-cubic-centimeters-per-second. □

10.4 For next time

For next time, you should be able to do all the exercises in today's lecture. This involves carefully applying rules of derivatives to known expressions, and understanding what numbers are needed to determine a rate of change that depends on other quantities.

10.5 Lab exercises

Exercise 10.5.1. Hiro is on an awful amusement park ride – the kind that swings back and forth, way up in the air. Hiro's height can be modeled by the following function:

$$h_{\text{Hiro}}(t) = 50 + 5\sin(\frac{\pi t}{30})$$

where time is measured in seconds and h is measured in meters. At time $t = 30$ seconds into the ride, Hiro stupidly drops his phone. The fall of his phone can be modeled by the function

$$h_{\text{phone}}(t) = 50 - 9.8(t-30)^2.$$

(a) Two seconds after Hiro dropped his phone, how far apart vertically are Hiro and his phone?

(b) Two seconds after Hiro dropped his phone, at what rate is the height between Hiro and his phone changing? What units are your answer?

Exercise 10.5.2. Two athletes, beginning at the same location, decide to run in perpendicular directions – A runs northward, while B runs eastward. Athlete A's position, as measured by the distance northward they have traveled, is modeled by the function

$$A(t) = 4\ln(t-1)$$

while Athlete B's position, measured by the distance eastward travelled, is modeled by the function

$$B(t) = 5t.$$

Both functions take in a time t as measured in seconds, and output distance as measured in meters.

(a) Write a function $d(t)$ that tells you the distance between athlete A and athlete B at time t seconds, as measured in meters. (Hint: Pythagorean theorem.)

(b) Write a function $v(t)$ that tells you the rate at which the distance between A and B is changing at time t, as measured in meters per second.

(c) At time $t = 2$ seconds into the run, how quickly is the distance between A and B growing, in meters per second?

Exercise 10.5.3. A culture of bacteria is growing on a petri dish. At any time t, the bacteria are taking up a circular region on the petri dish, and the radius of this region is modeled by the following function:

$$r(t) = 8t$$

where t is in seconds and r is in micrometers.

At $t = 3$ seconds, how quickly is the area of the circular region changing, in units of micrometers-squared-per-second?

Exercise 10.5.4. A teardrop falls onto a lake, and the resulting ripple grows in radius. The radius can be modeled as a function of time as follows:

$$r(t) = 5e^{-3t}$$

where r is in centimeters and t is in seconds.

In terms of centimeters-squared-per-second, how quickly is the area enclosed by the ripple growing at $t = 2$ seconds?

Exercise 10.5.5 (An optimization problem)**.** You are running a subscription service, and your analysts have told you that the number of subscribers changes as a function of subscription cost as follows:

$$S(x) = 100000 + 20000(10 - x)$$

where x is the dollar amount cost of a yearly subscription (per subscriber), and S is the number of yearlong subscribers.

What should you make your yearly subscription costs to maximize revenue from subscribers?