Lab exercises: Concavity, local minima and maxima

Exercise 9.1.1 (Facts of life)**.** For the following, you might be able to have guesses at answers. But the point of the exercise is to see *why* those guesses are correct, too!

(a) Among all pairs of numbers (a, b) that add to 100, which pair(s) have the largest value of $a \times b$?

(b) Among all pairs of *positive* numbers (a, b) that multiply to 100, which pair(s) have the smallest value of $a + b$? Is there a pair with a largest value of $a + b$?

Exercise 9.1.2. Consider the ellipse $x^2 + 9y^2 = 1$. Among all the rectangles one can fit inside this ellipse, what is the rectangle of largest area? (Give its dimensions.)

Exercise 9.1.3 (Drugs)**.** Two drugs – drug A and drug B – are modeled to have a synergistic effect. If their dosages, as measured in milligrams, multiplies to 12 milligrams-squared, a desired outcome is achieved. However, drug A costs 100 dollars per milligram, and drug B costs 300 dollars per milligram.

What is the minimum cost at which one can achieve the desired outcome using these two drugs? Say how much of drug A and how much of drug B should be taken to achieve this cost.

Some solutions

Solution to 9.1.1. For (a), we are given $a + b = 100$ and we must maximize the function ab. Because we are only used to functions with one input variable, let's choose either a or b. We will choose a to be the variable we keep around; you can try with b if you like, for practice.

Then $b = 100 - a$, so the function we are trying to maximize is the function $a(100 - a)$. To maximize this function, we want to find its critical points. If we have a critical point with negative second derivative, we know we have found a local maximum. The derivative of this function (with respect to a) is given by

$$(a(100 - a))' = 100 - 2a$$

so the derivative is zero when $a = 50$. You can check that the second derivative is negative at this value of a (and in fact, for all a). So this is a local maximum. Because the function is concave down everywhere, we conclude that this local maximum is a global maximum.

For (b), we are given that $ab = 100$ and we must minimize or maximize $a + b$. In other words, we must study the function $a + \frac{100}{a}$. So let's look for its critical points. The derivative of this function is given by

$$1 - \frac{100}{a^2}$$

so we see that this equals zero when $a^2 = 100$. In other words, $a = 10$ or -10. The problem asks us to consider only positive values of a and b, so the only answer we keep is $a = 10$.

Is this a maximum or a minimum? Taking the second derivative of $a + \frac{100}{a}$, we find

$$\frac{200}{a^3}$$

which at $a = 10$ is a positive number. So the value $a = 10$ is a local *minimum* by the second derivative test; and because the above second derivative is always positive for positive values of a, the function is always concave up, so this local minimum is a global minimum. We conclude that $a = 10, b = 10$ is the pair of numbers that minimizes $a + b$.

The question also asks if there is a pair of numbers that maximizes $a + b$. The answer is no. Suppose that someone claims they have values (a_0, b_0) for which $a_0 + b_0$ is maximal among all numbers multiplying to 100. Well, try setting $a = (a_0 + b_0 + 1)$ and set $b = 100/a$. Then these values of a and b clearly have a larger sum than $a_0 + b_0$. □

Solution to 9.1.2. A rectangle with one corner at the origin, and with opposite corner at a point (x, y) has area xy. This is the function we want to maximize.

On the other hand, we know that $y^2 = (1 - x^2)/9$ because (x, y) has to be on the given ellipse. So the area can be written as a function of x:

$$x\sqrt{1 - x^2}/3.$$

We wish to find the critical points of this area function. The derivative is computed as:

$$\frac{1}{3}\left(x\frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right) = \frac{1}{3}\left(x\frac{-x}{\sqrt{1-x^2}} + \frac{1-x^2}{\sqrt{1-x^2}}\right) \tag{9.1.1}$$

$$= \frac{1}{3}\frac{1-2x^2}{\sqrt{1-x^2}} \tag{9.1.2}$$

This expression equals zero precisely when $1 - 2x^2$ equal zero; that is, when

$$x^2 = \frac{1}{2}.$$

So we find that the critical point occurs when $x = \frac{\sqrt{2}}{2}$. We should verify that the second derivative is negative at this value of x (to make sure it is a local maximum) – I will leave that to you.

To give the full dimensions, we must now find the y coordinate. Plugging back into the equation for the ellipse, we must solve for the value of y satisfying

$$\frac{1}{2} + 9y^2 = 1.$$

We find that the solution is

$$y^2 = \frac{1}{18}.$$

So $y = \frac{\sqrt{2}}{6}$. Thus, the dimensions of the rectangle are

$$\frac{\sqrt{2}}{2} \text{ by } \frac{\sqrt{2}}{6}$$

□

Solution to 9.1.3. Let A denote the dosage (in milligrams) of drug A, and likewise B. We are constrained by $AB = 12$ and we wish to minimize the cost

$$100A + 300B.$$

Again changing the cost function to depend on only one variable – say, B – we must minimize the function

$$100\frac{12}{B} + 300B.$$

The derivative (with respect to B) is

$$\frac{-1200}{B^2} + 300$$

and this equals zero when

$$B^2 = \frac{300}{1200} = \frac{1}{4}.$$

Note that the second derivative of the cost function is

$$\frac{2400}{B^3}$$

which is always positive for positive values of B, so the above critical point is not only a local minimum, but a global minimum.

To summarize, we achieve our minimal cost when

$$B = \frac{1}{2}.$$

It follows that $A = 24$. So the dosages should be 24 milligrams of drug A and 0.5 milligrams of drug B. The cost associated to this dosing is

$$100A + 300B = 2400 + 150 = 2550.$$

2,550 dollars. An expensive drug dosage. □