Lecture 8

Second derivatives, concavity, and minima/maxima

Remark 8.0.1. At certain times today, we'll talk about whether a function f does or does not have a derivative. This might be jarring, because so far we've only seen functions that *do* have derivatives. But for now, know that there are functions out there who do not have derivatives (an example is $f(x) = |x|$).

8.1 Second derivatives

Today, we will practice taking "second derivatives," and knowing when they are positive or negative.

Definition 8.1.1. The second derivative of f is *the derivative of the derivative*[1] of f. We denote the second derivative by

$$f'', \qquad \text{or} \qquad \frac{d}{dx}(\frac{d}{dx}f), \qquad \text{or} \qquad \frac{d^2}{dx^2}f, \qquad \text{or} \qquad \frac{d^2 f}{dx^2}. \tag{8.1.1}$$

Example 8.1.2. Let $f(x) = 3x^2 + x - 7$. Then the (first) derivative of f is

$$f'(x) = 6x + 1.$$

If we take the derivative of $f'(x)$, we end up with the second derivative of f:

$$f''(x) = 6.$$

[1] Yes, there are two appearances of the word "derivative"; this is not a typo.

Example 8.1.3. Let's find the second derivative of $f(x) = \ln(x)$. As defined above, we just need need to take the derivative twice. Let's take the first derivative:

$$f'(x) = \frac{1}{x}.$$

(This is something we learned in class.) Now let's take another derivative—for example, by using the quotient rule—to find

$$f''(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}.$$

That is, the second derivative of $\ln x$ is $-1/(x^2)$.

Exercise 8.1.4. Compute the second derivatives of the following:

(a) $f(x) = \sin(x)$.

(b) e^x

(c) e^{5x}.

(d) $f(x) = x^3 - 5x^2$.

If you know how to take derivatives, you know how to take second derivatives. So you see how our skills are building on each other—make sure you practice taking derivatives!

Exercise 8.1.5. Let $f(x) = x^2 - 2$. *Where is the second derivative positive?*

Possible solution. Let's find the second derivative. We see that

$$f'(x) = 2x$$

so

$$f''(x) = 2.$$

So the second derivative is always 2, meaning the second derivative is positive *everywhere.* □

Exercise 8.1.6. Let $f(x) = x^3 - 3x^2 + 3$. *Where is the second derivative positive?*

Possible solution. Let's find the second derivative. We see that

$$f'(x) = 3x^2 - 6x$$

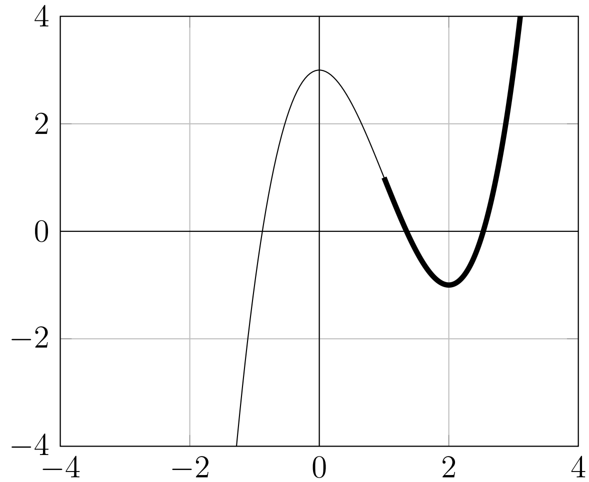
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 6x - 6.$$

So the second derivative is positive when $6x - 6$ is positive. This happens exactly when $6x > 6$—that is, when $x > 1$.

As a bonus: The second derivative is negative when $6x < 6$—that is, when $x < 1$.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:



□

Exercise 8.1.7. Let $f(x) = x^4 - 24x^2 + 50$. Where is the second derivative positive?

Possible solution. Let's find the second derivative. We see that

$$f'(x) = 4x^3 - 48x$$

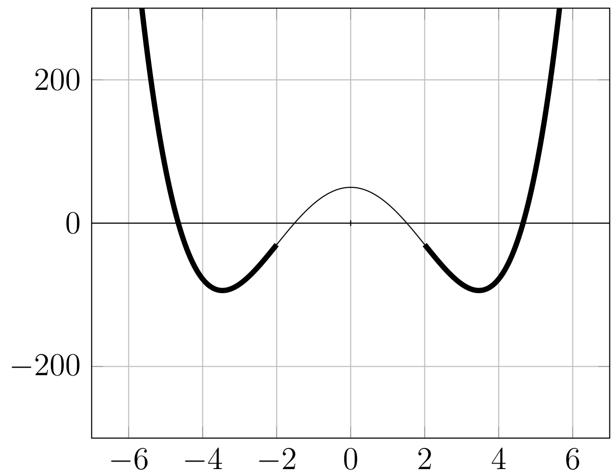
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 12x^2 - 48.$$

So the second derivative is positive when $12x^2 - 48$ is positive. This happens exactly when $12x^2 > 48$—that is, when $x^2 > 4$. But $x^2 > 4$ exactly when $x < -2$ or $x > 2$.

As a bonus: The second derivative is negative when $x^2 < 4$—that is, when x is between -2 and 2.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:



□

Exercise 8.1.8. Let $f(x) = 3\sin(x)$. Where is the second derivative positive?

Possible solution. Let's find the second derivative. We see that

$$f'(x) = 3\cos(x)$$

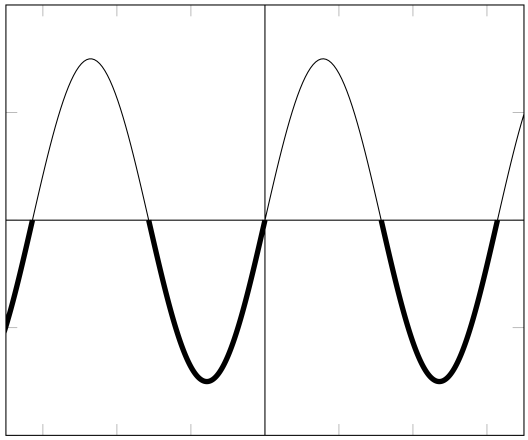
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = -3\sin(x)$$

So the second derivative is positive when $-3\sin(x)$ is positive. This happens exactly when $\sin(x)$ is negative. And based on our trigonometry knowledge from precalculus, we know that this happens when

- x is between π and 2π,
- x is between 3π and 4π,
- x is between $-\pi$ and 0,
- x is between -3π and $-\pi$,
- $\ldots$.

Below is a graph of $f(x)$. I have shaded in **bold** the part of the graph where the second derivative is positive:



$\square$

8.2 Concavity

The point I want to make with these pictures is that *the value of the second derivative gives us some idea of what the graph looks like.* (Though not a complete picture.)

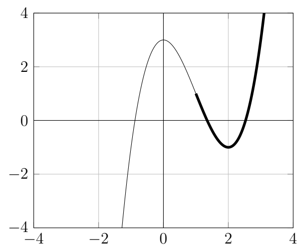
Intuition: On the regions where the second derivative is positive, the graph of f looks like a *portion* of an "upright bowl." Some students have described this as "opening upward" as well.

Conversely, when the second derivative is negative, the graph of f looks like a portion of an "upside-down bowl." But we have technical names, too. From now on, you are expected to know the following terminology:

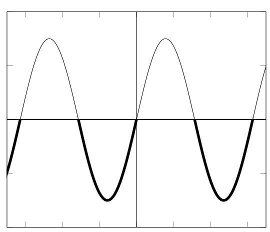
Definition 8.2.1 (Concavity)**.** We say that f is *concave up* at x if $f''(x) > 0$. We say that f is *concave down* at x if $f''(x) < 0$.

You have seen examples of graphs with positive second derivative. Here are some examples, with the positive-second-derivative regions shaded in *bold.* In other words, bolded are the regions where the graph is concave up. In the following examples, the unbolded regions are where the graph is concave down.

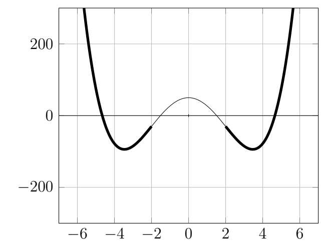
1. $f(x) = x^3 - 3x^2 + 3$:



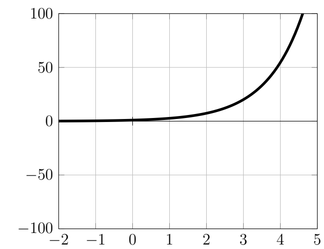
2. $f(x) = 3\sin(x)$:



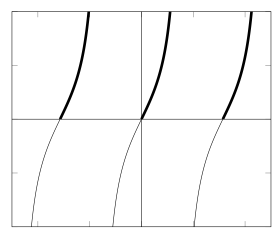
3. $f(x) = x^4 - 24x^2 + 50$:



4. $f(x) = e^x$:



5. $f(x) = \tan(x)$:

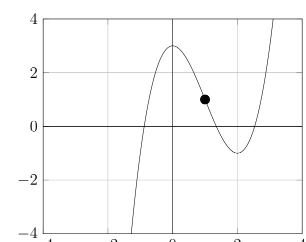


8.3 Inflection points

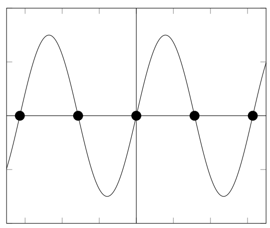
Definition 8.3.1. If $f''(x) = 0$, and the concavity of f *changes* at x, we say that x is an *inflection point*.

Example 8.3.2. Here are some examples of functions and their graphs, with their inflection points labeled.

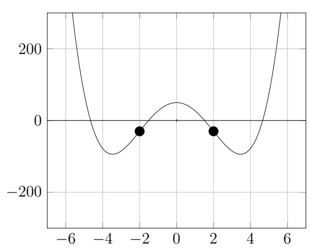
1. $f(x) = x^3 - 3x^2 + 3$:



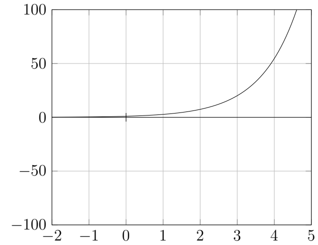
2. $f(x) = 3\sin(x)$:



3. $f(x) = x^4 - 24x^2 + 50$:

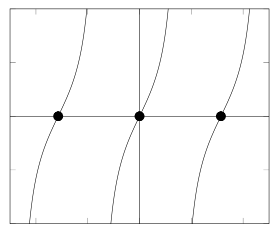


4. $f(x) = e^x$:

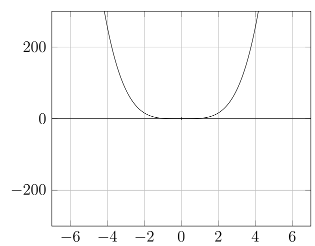


(No inflection points.)

5. $f(x) = \tan(x)$:



6. $f(x) = x^4$:



(No inflection points, even though $f''(x) = 0$ at $x = 0$.)

Expectation 8.3.3. Based on looking at a graph, you are expected to be able to identify inflection points—an inflection point is a place at which a function switches concavity (from up to down, or from down to up).

Exercise 8.3.4. Now we're going to try to learn something about a function by knowing its derivative and second derivative. You can hunt for examples on the previous pages of this packet. Or, you can try understanding $f(x) = x^2$ and $f(x) = -x^2$.

1. Can you find an example of a function f, and a point x, where $f'(x) = 0$ and f is concave up at x? What does the function f look like near x? How does the value of f at x compare to the value of f at nearby points?

2. Can you find an example of a function f, and a point x, where $f'(x) = 0$ and f is concave *down* at x? What does the function f look like near x? How does the value of f at x compare to the value of f at nearby points?

Exercise 8.3.5. Tell me the second derivatives of the following functions:

(a) $x^3 - 3x^2 + x$

(b) $4x^2 + 3x - 2$

(c) e^{7x}

(d) $\sin(x)$

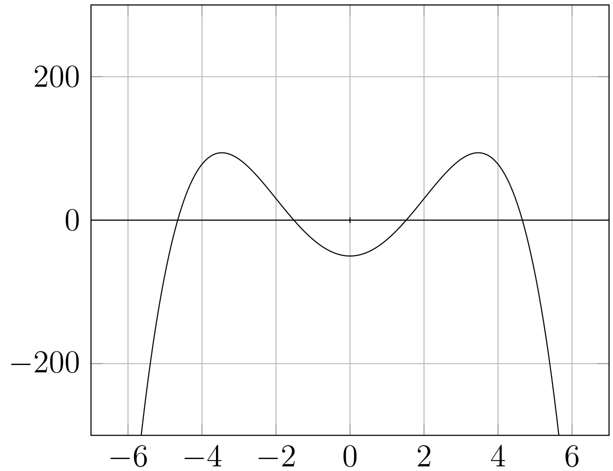
Exercise 8.3.6. Tell me where the following functions are concave up:

(a) $x^3 - 3x^2 + x$

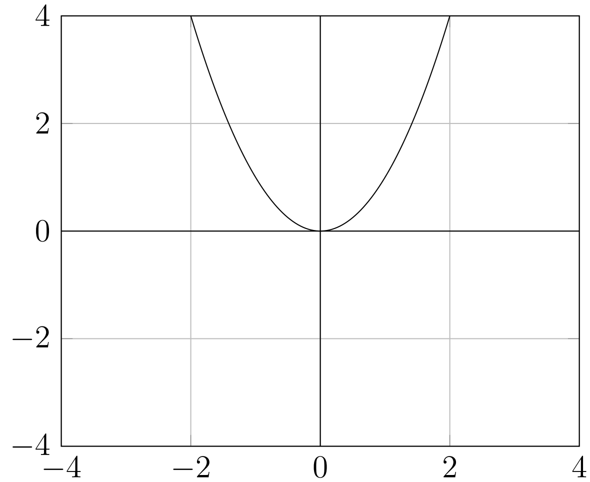
(b) $4x^2 + 3x - 2$

(c) e^{7x}

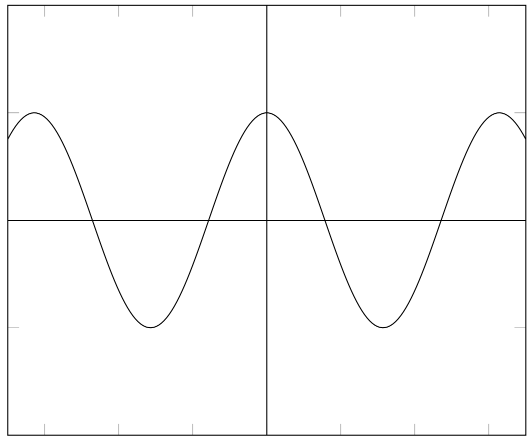
Exercise 8.3.7. For each of the functions $f(x)$ below, Shade in **bold** where the graph of the function has positive second derivative. (You are provided the graph of $f(x)$.) *Draw a dot* at every inflection point.



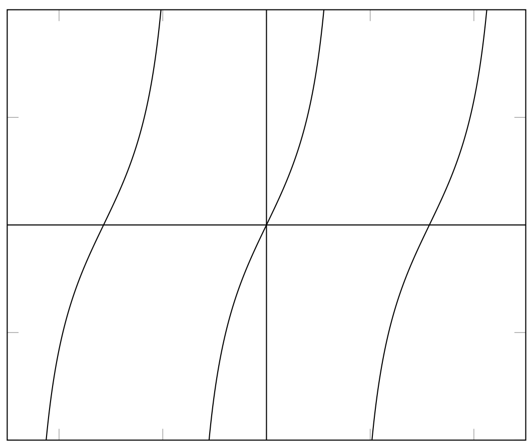
(a) $f(x) = -x^4 + 24x^2 - 50.$



(b) $f(x) = x^2.$



(c) $f(x) = \cos(x)$.



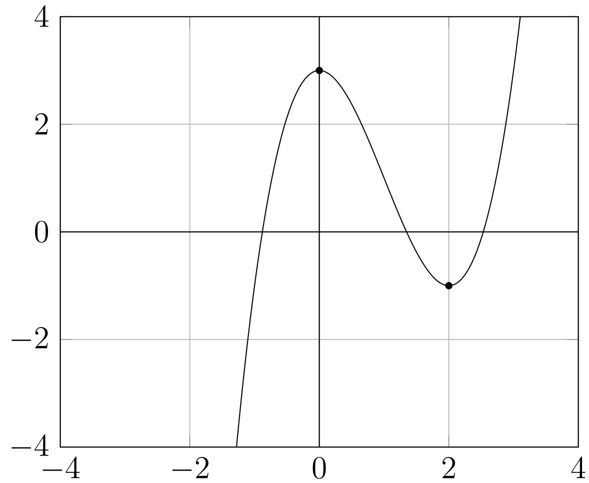
(d) $f(x) = \tan(x)$.

8.4 Local extrema (minima and maxima)

Now we're going to study places where graphs look maximal or minimal. (That is, where functions attain their highest and lowest points – "locally.")

8.4.1 Local maxima and local minima

Let's study the example of $f(x) = x^3 - 3x^2 + 3$:



Just by looking at the graph, we can see the two points where the derivative of f is zero (i.e., the two points where the tangent lines are horizontal). They roughly occur at $x = 0$ and $x = 2$. (And you can prove that they *exactly* occur there if you do out the math—that is, if you solve the equation $f'(x) = 0$ for x.)

We see that at $x = 0$, the function is concave down. Moreover, it looks like $f(0)$ is the *biggest* value that f achieves near $x = 0$. We will call such a point a **local maximum**. (That is, $x = 0$ is a local maximum.)

And at $x = 2$, we see that the function is concave up. Moreover, it looks like $f(2)$ is the *smallest* value that f achieves near $x = 2$. We call such a point a **local minimum** (so $x = 2$ is a local minimum). A point is called a **local extremum** if it is either a local maximum or a local minimum.

The plural form of maximum is "maxima." Likewise, the plural form of minimum is minima, and the plural form of extremum is extrema.

Remark 8.4.1. Finding local maxima and minima have huge important in real life. For example, you could imagine $f(x)$ to measure a model for profit given a particular input x. Then you'd like to optimize to maximize profit within a feasible input range of values of x. In other words, you may want to find *local maxima* of the function f.

Likewise, when designing a particular system, $g(x)$ may measure the amount of risk, or the probability of failure, given some input parameter x. Then you would like to optimize to reduce risk as much as possible within some feasible input values of x. In other words, you may want to find *local minima* of the function g.

Your intuition might tell you that wherever there is a local maximum or a local minimum, the graph should have a "trough" or a "crest." In particular, the derivative should be zero there! This is true so long as the function is differentiable:

Theorem 8.4.2. If f has a derivative, and if x is a local minimum or a local maximum, then $f'(x) = 0$.

Warning 8.4.3. These minima and maxima are called "local." This is because if x is a local minimum, it may not be true that $f(x)$ is the "minimum" value that f can take!

In the example above of $f(x) = x^3 - 3x^2 + 3$, we see that $f(x)$ can take as negative a value as it wants, so f has no "absolute minimum." Likewise, $f(x)$ can take as positive value as it wants, so f has no "absolute maximum." It only has a "local" minimum at $x = 2$, where the value of $f(2)$ is smaller than the value at all *neighboring* points (i.e., all nearby points).

8.5 Critical points

So it will be important for us to find x for which f' vanishes. Such special points have a name:

Definition 8.5.1. Let f be a function. We say that x is a *critical point* of f if $f'(x) = 0$.

Exercise 8.5.2. Verify the following:

(a) If $f(x) = 5$, every point is a critical point.

(b) If $f(x) = 3x$, f has no critical points.

(c) If $f(x) = x^2$, $x = 0$ is a critical point.

(d) In fact, zero is a critical point for $f(x) = x^3$ and for $f(x) = x^4$, and so forth.

Warning 8.5.3. Not all critical points are local extrema. (For example, look at the critical point of $f(x) = x^3$.)

Warning 8.5.4. If f does not have a derivative, not all local extrema are critical points. Consider the example of $f(x) = |x|$. This has a minimum at $x = 0$, but f does not have a derivative there (as we have seen before).

8.6 The second derivative test

The following is called the **second derivative test** for finding local maxima and local minima. You in fact discovered it in Exercise 8.3.4.

Theorem 8.6.1 (The second derivative test)**.** Let f be a function that has a derivative, and x a number. Suppose that $f'(x) = 0$ and $f''(x) > 0$. Then f has a *local minimum* at x.

Suppose that $f'(x) = 0$ and $f''(x) < 0$. Then f has a *local maximum* at x.

This helps us draw f: We know that f looks like a hump/hilltop/crest where f has a local maximum. And we know that f looks like a bowl/trough/nadir where f has a local minimum.

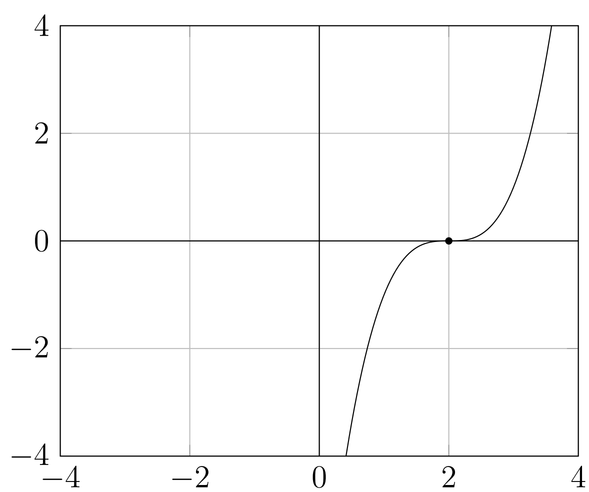
In simple terms, the theorem states the following: If the tangent to f is flat at x, and f looks like (part of) an upward-opening bowl at x, then x must be the *bottom* of the bowl (hence a local minimum).

Likewise, if the tangent to f is flat at x, and if f looks like (part of) a downward-opening bowl at x, then x must be the *top* of that bowl (hence a local maximum).

8.6.1 The second derivative test can be inconclusive

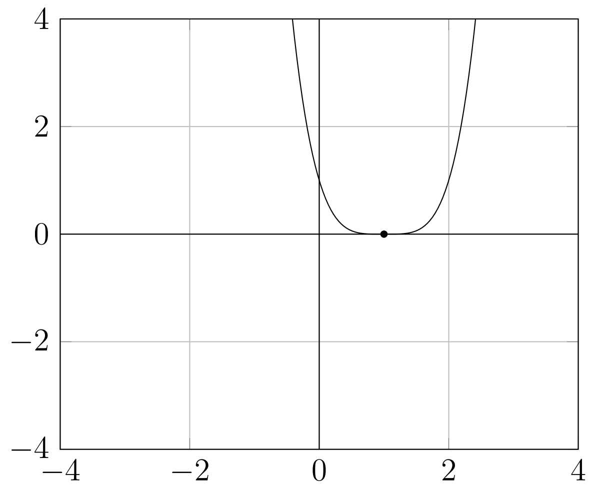
If $f'(x) = 0$ and $f''(x) = 0$, we *do not know* whether we have a local maximum or minimum (or neither)! Here are two examples:

Example 8.6.2. Consider $f(x) = (x-2)^3$. Then—check this!—$f'(2) = 0$ and $f''(2) = 0$. Below is a graph of $f(x)$:



This is a strange example, but it is a great one. As you can see, the graph does have "flat" tangent line at $x = 2$, but $x = 2$ is neither a local maximum nor a local minimum—I can immediately get larger than $f(2) = 0$ by moving right, or immediately get smaller than $f(2) = 0$ by moving left.

Example 8.6.3. Here is the example of $f(x) = (x - 1)^4$. We can check easily that $f'(1) = 0$ and $f''(1) = 0$.



As we can see from the picture, we have a *local minimum* at $x = 1$.

The conclusion from the above two examples is: If the hypotheses of the second derivative test are not met, we have to do more work to determine whether we have a local minimum or maximum.

8.6.2 Exercises

Tell me the critical points, the local maxima, the local minima, and the the critical points where the second derivative test is inconclusive, for the following functions:

(a) $f(x) = x^3$

(b) $f(x) = x^3 - 3x^2 + 7$

(c) $f(x) = x \ln x$

(d) $f(x) = xe^x$.

8.7 For next time

For next class, I expect you to be able to complete all the exercises from today.