Lecture 7

The product rule and quotient rule

Today, we'll see how to take the derivative of a *product* of two functions.

7.1 The Leibniz rule aka product rule

The Leibniz rule (also known as the product rule). If f and g have derivatives at x, then so does the product fg, and the derivative is computed as follows:

$$\left(\frac{d}{dx}(f \cdot g)\right)(x) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)).$$

Written another way,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Remark 7.1.1. It's up to you which name you prefer for this rule—Leibniz rule or product rule—just be aware that both names are in common use. (So you know what other people are talking about if they mention it.)

Also, remember that in math, the word *product* means the multiplication of something; more specifically, something is called a product if it is being viewed as the multiplication of two things.

For example, 15 is the product of 3 and 5. You can also multiply functions, so that $3x\cos(x)$ is the product of $3x$ and $\cos(x)$.

7.1.1 Interpretation of the product rule

First, the product rule applies to functions that arise as products of two other functions. What kinds of situations arise in real life where we care about functions that are products? Well, a lot.

For example, how much electricity does a town need? How much will it need in the future? One way to estimate this is to think about how many people the town will have at a future time t – let's call this $f(t)$ – and how much energy the town will be using per person at that time; let's call this $g(t)$. These are reasonable things to estimate – we obviously want to predict future populations of towns, and we know that each person is using less energy as appliances and transportation technology become more energy efficient.

Then the total amount of electricity that the town uses at time t is modeled by the product

$$f(t)g(t).$$

Remark 7.1.2. You can imagine that functions that depend on two other functions in this way show up all the time. Other examples might have f representing the density of a material and g representing its volume; then fg will tell us the total weight of the material. One could imagine that $f(t)$ models the number of cashiers at a fast food chain t years from now, and $g(t)$ models the hourly wage of a cashier t years from now. (As you know, hourly wages are supposed to change – in fact, increase – from year to year.) Then $f(t)g(t)$ tells us the total amount of money, per hour, the fast food chain is spending on cashier wages.

The derivatives of these functions are all important. For example, in the last example of wages, the fast food company better make sure that it can increase its revenue at a rate that keeps up with the rate at which cashier wages are going up.

Keeping up with the example of electricity use of a town, let's compute the rate at which energy use is changing in the town:

$$(f(t)g(t))' = f'(t)g(t) + f(t)g'(t).$$

The lefthand side is what we're after: It's the derivative of energy use. It has units of, say, kilowatt-hours per year. It's roughly a measure of the rate at which energy use is changing in the city. The righthand side is what the product rule tell us – and it's worth interpreting.

The rate at which energy use changes doesn't depend just on f' and g', which are the rates at which the population and the energy use per person is changing. It also depends on the actual values of f and g – that is, on how many people there actually are, and on how much energy each person is using.

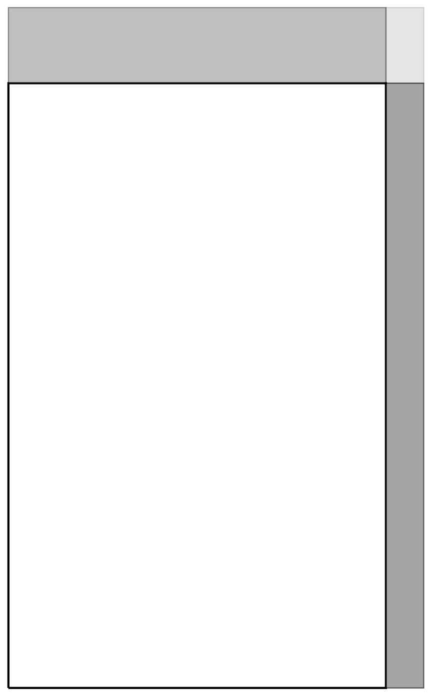
In other words, $(fg)'$ cannot be predicted or calculated based purely on f' and g'. You need to know the *values* of f and g as well.

7.1.2 Intuition for the product rule

The product rule is one of the weirdest rules the first time you see it. Why is it the way that it is?

Here is a simple example:

Let's say a rectangle is expanding – maybe it's a plot of land occupied by ants, and the ants are taking over more and more land in the shape of a rectangle. Now suppose that we know how big this rectangle is, and we also know how much the width and height of this rectangle grow over a small period of time:



In the picture, the shaded regions indicate the growth – so the width grew by a tiny bit (as indicated by the darker shaded region) and the height grew a bit more (as indicated by next darkest shaded region). How much new area have we tagged on to the original white rectangle? Obviously, the area of the two darker new regions don't depend just on the increase in width, or just the increase in height – the areas of the shaded rectangles depend on the width and height of the *original* white rectangle. In other words, on the actual values of width and height, not just how they are changing.

(Remark: The tiny, lightest-shade rectangle in the upper-right corner is also an amount by which area is changing. It's a little subtle that this term doesn't appear in the product rule, but if you think about units, you might be happy: The light tiny rectangle in the upper-right would correspond to a term that looks like "$f'g'$", but this would have units of area per time-squared, not area per time. So a term like this shouldn't pop up in the product rule.)

7.1.3 Computing with the product rule

Exercise 7.1.3. Compute the derivatives of the following functions:

(a) $x^2 \sin(x)$

(b) $x^3 \cos(x)$

(c) $(3x^2 + x)(x - 3)$

Possible solutions. (a) Let's compute the derivative of $x^2 \sin(x)$. We have

$$\frac{d}{dx}(x^2 \sin(x)) = \left(\frac{d}{dx}(x^2)\right) \cdot \sin(x) + x^2 \cdot \left(\frac{d}{dx}(\sin(x)\right) \tag{7.1.1}$$

$$= 2x \cdot \sin(x) + x^2 \cos(x). \tag{7.1.2}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^2 and of the derivative of sin.

(b) Let's compute the derivative of $x^3 \cos(x)$. We have

$$\frac{d}{dx}(x^3 \cos(x)) = \left(\frac{d}{dx}(x^3)\right) \cdot \cos(x) + x^3 \cdot \left(\frac{d}{dx}(\cos(x)\right) \tag{7.1.3}$$

$$= 3x^2 \cdot \cos(x) - x^3 \sin(x). \tag{7.1.4}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^3 (the power law) and of the derivative of cos.

(c) Let's compute the derivative of $(3x^2 + x)(x - 3)$. This won't involve any sin or cos.

There are two ways to do this. One way to do this is by using the Leibniz rule:

$$\frac{d}{dx}\big((3x^2 + x)(x - 3)\big) = \frac{d}{dx}(3x^2 + x) \cdot (x - 3) + (3x^2 + x) \cdot \frac{d}{dx}(x - 3) \tag{7.1.5}$$

$$= (3 \cdot 2x + 1) \cdot (x - 3) + (3x^2 + x) \cdot 1 \tag{7.1.6}$$

$$= (6x + 1) \cdot (x - 3) + (3x^2 + x)1 \tag{7.1.7}$$

$$= 6x^2 - 17x - 3 + 3x^2 + x \tag{7.1.8}$$

$$= 9x^2 - 16x - 3. \tag{7.1.9}$$

Another way is to first multiply the factors together, and then use the addition and power rules:

$$\frac{d}{dx}\Big((3x^2+x)(x-3)\Big) = \frac{d}{dx}\Big(3x^3-9x^2+x^2-3x\Big) \tag{7.1.10}$$

$$= \frac{d}{dx}\Big(3x^3-8x^2-3x\Big) \tag{7.1.11}$$

$$= 3\cdot 3x^2 - 2\cdot 8x - 3 \tag{7.1.12}$$

$$= 9x^2 - 16x - 3. \tag{7.1.13}$$

□

Exercise 7.1.4. A rectangular rubber sheet is expanding in the sun. Its width at time t seconds is given by $3t$ centimeters. Its length at time t is given by e^t centimeters.

(a) Write a function telling us the area of this rectangular rubber sheet at time t seconds. What are the units of this function?

(b) Write a function telling us the *rate* at which the area of this rubber sheet is changing at t seconds. What are the units of this function?

(c) How quickly is the area changing at $t = \ln 3$ seconds?

Exercise 7.1.5. An inflated balloon has just been placed in Hiro's freezer. It is shrinking due to the cold temperatures, just like Hiro's bank account. (Non sequitor.)

Using a camera inside the freezer, Hiro observes that the volume of this ballon at time t seconds is given as follows:

$$V(t) = 10e^{-t}$$

in cubic centimers. He does not know what the density at time t seconds is, but he decided to call the density at time t

$$D(t)$$

in kilograms per cubic centimeters.

1. In terms of $V(t)$ and $D(t)$, what is the mass of this balloon, in kilograms, at time t seconds?

2. Hiro wants to know the rate at which the mass of the balloon is changing at time t seconds. In terms of V, D, V' and D', what is this rate of change?

3. What are the units of this rate of change?
4. Hiro's freezer has a scale in it for some reason. At time $t = 2$ seconds, the scale tells Hiro that the balloon weighs 0.01 kilograms. Based on his observations, Hiro also happens to know that the balloon's weight is changing at a rate of -0.001 kilograms per second. (It's getting lighter.) Does Hiro know enough to determine the rate at which the balloon's density is changing at time $t = 2$?
5. If so, what is that rate?
6. Regardless, what are the units of this rate?

Exercise 7.1.6. Compute the derivatives of the following functions:

1. $-\cos(x)$
2. $x - \cos(x)$
3. $-x\cos(x)$
4. $-x^2\cos(x)$.
5. $\sin(x)^2$.
6. $\sin(x)\cos(x)$.
7. $x^3 + 3x - 2$.
8. $(x-3)(x-2)$.
9. $(x^2-1)(3x-1)$.

Exercise 7.1.7. What is the slope of the tangent line to the graph of $f(x) = x\cos(x)$ at $x = \pi/4$?

7.2 Quotient Rule

Recall that in math, the word *quotient* refers to something obtained by division. For example, 3 is the quotient of 15 by 5. Likewise, the function

$$\frac{3x}{\cos x}$$

is the quotient of $3x$ by $\cos x$.

Now, we will finally know how to take derivatives of quotients:

Theorem 7.2.1 (The quotient rule.)**.** Whenever f and g are differentiable at x, and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}.$$

Put another way,

$$\frac{d}{dx}\left(\frac{f}{g}\right)(x) = \frac{\frac{df}{dx}(x)g(x) - \frac{dg}{dx}(x)f(x)}{g(x)^2}.$$

Written yet another way, the quotient rule is

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

Warning 7.2.2. The hardest part about the quotient rule is remembering the order of things in the numerator:

$$f'g - fg'.$$

Note that the positive term is $f'g$, and the negative term is fg'. Some people prefer to remember the numerator as

$$f'g - g'f,$$

which is the same thing.

Do *not* make the mistake of writing something like $g'f - f'g$ in the numerator. This is the wrong answer.

Exercise 7.2.3. Find the derivatives of the following functions using the quotient rule.

(a) x^{-1}

(b) x^{-3}

(c) x^{-101}

(d) $\frac{1}{x^3+3}$

(e) $\frac{x}{x^2+3}$

(f) $\frac{x-1}{x+1}$

(g) $\frac{\sin(x)}{\cos(x)}$

(h) $\frac{\cos(x)}{\sin(x)}$

(i) $\frac{e^x}{x}$

(j) $\frac{\ln(x)}{x}$

Exercise 7.2.4. Find the derivative of e^{-x} in at least two different ways, one of them being the quotient rule, and another way using only what you learned before this class.

Exercise 7.2.5. Find the derivative of x^{-5} in at least two different ways, one of them being the quotient rule, and another way using the product rule cleverly. (Hint: $x^{-5} \cdot x^5 = 1$.)

Example 7.2.6. We know the derivative of $x^3 + 3$. Can we compute the derivative of

$$\frac{1}{x^3+3}?$$

Let $f(x) = 1$ and $g(x) = x^3 + 3$. Then we know $g' = 3x^2$. So the formula from the Lemma tells us:

$$\frac{d}{dx}(\frac{1}{g(x)}) = \frac{-g'(x)}{g(x)^2} \tag{7.2.1}$$

$$= \frac{-(3x^2)}{(x^3+3)^2} \tag{7.2.2}$$

$$= \frac{-3x^2}{(x^3+3)^2}. \tag{7.2.3}$$

I won't simplify this fraction any further; though you could multiply out the bottom if you like.

Example 7.2.7 (The derivative of tangent)**.** Let's compute the derivative of

$$\frac{\sin(x)}{\cos(x)}.$$

(This is known, of course, as the *tangent* function.) Follow along:

$$\left(\frac{\sin}{\cos}\right)' = \frac{\sin' \cdot \cos - \sin \cdot \cos'}{\cos^2} \tag{7.2.4}$$

$$= \frac{\cos \cdot \cos - \sin \cdot (-\sin)}{\cos^2} \tag{7.2.5}$$

$$= \frac{\cos^2 + \sin^2}{\cos^2} \tag{7.2.6}$$

$$= \frac{1}{\cos^2} \tag{7.2.7}$$

$$= \sec^2 . \tag{7.2.8}$$

The first equality is using the quotient rule. The equality (7.2.7) follows from the identity $\sin^2 + \cos^2 = 1$. (You will be expected to know this identity; it's from trigonometry!) The last equality (7.2.8) follows from the definition of secant: $\sec = 1/\cos$.

We have proven:

$$\frac{d}{dx} \tan = \sec^2 .$$

Equivalently,

$$\tan'(x) = \sec(x)^2.$$

Example 7.2.8. Find the derivative of

$$\frac{x^2 - 3}{x^3 + 1}.$$

We use the quotient rule:

$$\left(\frac{x^2 - 3}{x^3 + 1}\right)' = \frac{(x^2 - 3)' \cdot (x^3 + 1) - (x^2 - 3) \cdot (x^3 + 1)'}{(x^3 + 1)^2} \tag{7.2.9}$$

$$= \frac{2x \cdot (x^3 + 1) - (x^2 - 3) \cdot 3x^2}{(x^3 + 1)^2} \tag{7.2.10}$$

$$= \frac{2x^4 + 2x - 3x^4 + 9x^2}{(x^3 + 1)^2} \tag{7.2.11}$$

$$= \frac{-x^4 + 9x^2 + 2x}{(x^3 + 1)^2}. \tag{7.2.12}$$

7.2.1 Bonus: Proof of the quotient rule

The quotient rule's proof requires a few steps. We often use the word "lemma" to refer to a step we use to prove an important result.

Lemma 7.2.9. Whenever $g(x) \neq 0$ and g has a derivative at x, we have:

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-\frac{dg}{dx}(x)}{g(x)^2}.$$

Put another way,

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}.$$

Put yet another way,

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g(x)^2}.$$

Here is a proof of this fact:

Proof. Let's begin by noticing that

$$1 = g(x) \cdot \frac{1}{g(x)} \qquad \text{(whenever } g(x) \neq 0\text{)}.$$

Because the function on the right is equal to the function on the left (they are both constant functions), their derivatives will be equal. Taking the derivatives of both sides, we have:

$$0 = \frac{d}{dx}(g(x) \cdot \frac{1}{g(x)}).$$

Using the product rule on the righthand side, we find:

$$0 = g'(x) \cdot \frac{1}{g(x)} + g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Moving a term over to the left, we find

$$-g'(x) \cdot \frac{1}{g(x)} = g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Dividing by $g(x)$ and simplifying, we find:

$$\frac{-g'(x)}{g(x)^2} = \frac{d}{dx}\frac{1}{g(x)} \qquad \text{when } g(x) \neq 0.$$

This proves the lemma! □

Proof of the quotient rule. Let's note that

$$\frac{f}{g} = f \cdot \frac{1}{g}.$$

So we can compute

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) && (7.2.13)\\
&= (\frac{d}{dx}f)(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{d}{dx}\left(\frac{1}{g(x)}\right) && (7.2.14)\\
&= \frac{\frac{df}{dx}(x)}{g(x)} + f(x) \cdot \left(\frac{-\frac{dg}{dx}(x)}{g(x)^2}\right) && (7.2.15)\\
&= \frac{g(x)\frac{df}{dx}(x)}{g(x)^2} + \frac{-f(x)\frac{dg}{dx}(x)}{g(x)^2} && (7.2.16)\\
&= \frac{\frac{df}{dx}(x)g(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2}. && (7.2.17)
\end{aligned}$$

The most important thing to note is that to conclude the equality in (7.2.15), we used Lemma 7.2.9 from above.

The beginning and the end of this string of inequalities is exactly what the quotient rule says. □

Remark 7.2.10. If the notation in the above proof is unappealing, here is a proof using only the "prime" notation:

$$\begin{aligned}
\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' && (7.2.18)\\
&= f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' && (7.2.19)\\
&= \frac{f'}{g} + f \cdot \left(\frac{-g'}{g^2}\right) && (7.2.20)\\
&= \frac{gf'}{g^2} + \frac{-fg'}{g^2} && (7.2.21)\\
&= \frac{f'g - fg'}{g^2}. && (7.2.22)
\end{aligned}$$

Again, the equality in (7.2.20), follows from Lemma 7.2.9 from above.

7.3 For next lecture

For next lecture, I expect you to be able to do any of the exercises in Exercise 7.1.6 and Exercise 7.2.3.