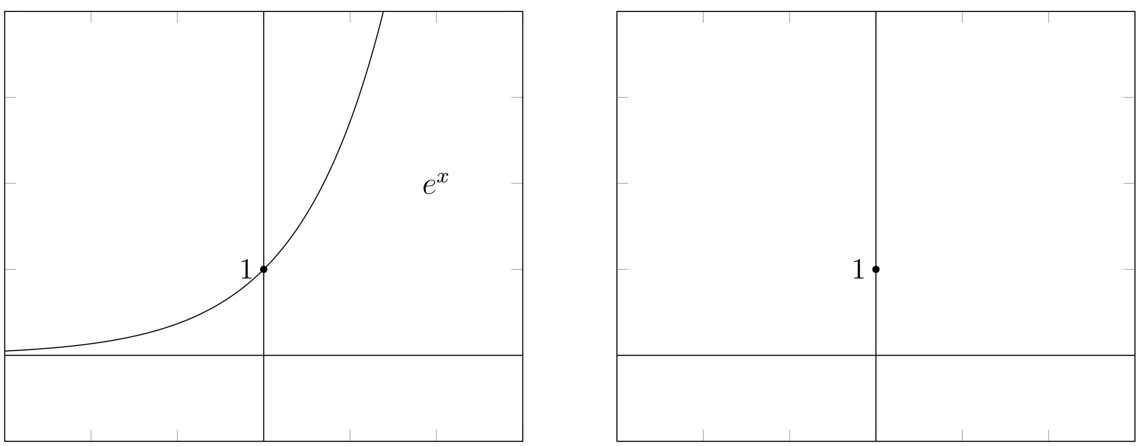
Lecture 6

Derivatives of exp, ln, and other inverse functions

6.1 Drawing derivatives of e^x

Exercise 6.1.1. Below on the left is the graph of $f(x) = e^x$.



Let me tell you the following fact: The derivative of e^x at $x = 0$ is 1. (In fact, the value of e^x at $x = 0$ is 1 also.)

(a) Based on this, draw the derivative of e^x on the right.

(b) How does your drawing compare to the graph of e^x?

6.2 e^x is its own derivative

In fact, we have the following theorem:

Theorem 6.2.1 (Derivative of e^x)**.** The derivative of e^x is itself. That is,

$$(e^x)' = e^x.$$

Put another way,

$$\frac{d}{dx}(e^x) = e^x.$$

How cool is that? There's a function that is its *own* derivative!

Exercise 6.2.2. Find the derivative of e^{3x}

Possible solution. Let's find the derivative of e^{3x}. We have

$$\begin{aligned} \frac{d}{dx}(e^{3x}) &= \frac{d}{dx}(3x) \cdot \frac{d(e^x)}{dx}(3x) && (6.2.1) \\ &= 3 \cdot e^{3x}. && (6.2.2) \end{aligned}$$

We have used the chain rule in the first line. If you're confused by it, it may be worthwhile to write this out step-by-step. Let's let $f(x) = e^x$ and $g(x) = 3x$. Then $e^{3x} = f(g(x))$. Thus

$$\begin{aligned} \frac{d}{dx}(e^{3x}) &= \frac{d}{dx} f(g(x)) && (6.2.3) \\ &= f'(g(x)) \cdot g'(x) && (6.2.4) \\ &= f'(3x) \cdot g'(x). && (6.2.5) \end{aligned}$$

(The second equality is due to the chain rule.) But we know that $f'(x) = e^x$ by Theorem 6.2.1, and we know $g'(x) = 3$ from previous lectures. Hence we can continue:

$$f'(3x) \cdot g'(x) = e^{3x} \cdot 3 = 3e^{3x}.$$

□

Exercise 6.2.3. Fix a real number B. Show me that the derivative of

$$f(x) = e^{Bx}$$

equals

$$Bf(x).$$

More generally, if you have another real number A, let

$$g(x) = Ae^{Bx}.$$

(For example, if you choose $A = 3$ and $B = 5$, you would have $3e^{5x}$. The previous example is when $A = 1$.) Explain why

$$g'(x) = Bg(x).$$

Application 6.2.4. This kind of behavior is incredibly important for *modeling*. For example, how fast is a population growing? In ideal circumstances, the more individuals there are in a population, the faster we expect the population to grow. Better yet, we might expect that the rate of population growth is *proportional* to the population itself! (Note that "being proportional to" is a far more precise relationship than "the bigger the population, the faster the growth".)

That's exactly what Exercise 6.2.5 tells us about $g(x) = Ae^{Bx}$. We see that g' is proportional to g (with proportional constant B). So for example, x could model time, while $g(x)$ could model the population at time x.

By the way, why might $g(x)$ be a *bad* model for population growth? For what kinds of situations might $g(x)$ be a *good model*? In those situations, what might A and B represent?

Exercise 6.2.5. Find the derivative of $f(x) = 5^x$. Hints: Remember that $5 = e^{\ln 5}$, remember the basic rules for dealing with exponents, and use the chain rule.

Exercise 6.2.6. Your friend is excited about the idea that $f(x)$ could equal $f'(x)$ and looks for more examples that looks like e^x. They try $f(x) = 5^x$, and are disappointed that $f'(x)$ does not equal $f(x)$.

Is it possible to find any number k—other than e—so that if $f(x) = k^x$, then $f'(x) = f(x)$?

Remark 6.2.7. Isn't e special?

Exercise 6.2.8. Now that you know the derivative of $g(x) = e^x$, can you figure out the derivative of $f(x) = \ln x$?

Hint: What is $g \circ f$? What if you try computing $(g \circ f)'$ using the chain rule, too?

6.3 Review of inverse functions

Let f be a function. Here's a question: Given a value of f, can we always determine which x it came from?

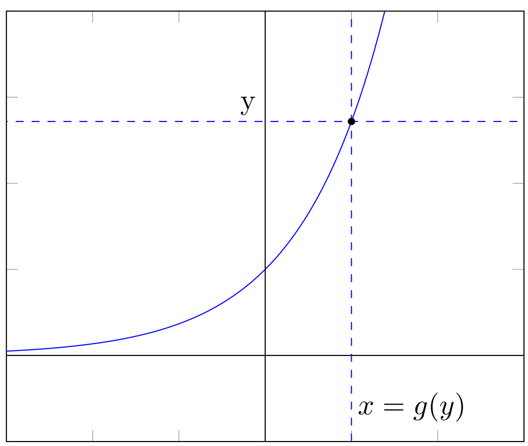
Example 6.3.1. Here are some examples:

1. If $f(x) = 3x$, and if someone tells you that f takes the value 12, you know exactly where: x must equal 4. In fact, in general, if f takes value y, you know the original x is $y/3$.

2. If $f(x) = 2^x$, and if someone tells you that f takes the value 8, you know exactly where: x must equal 3. In fact, in general, if f takes value y, you know f does so at $\log_2 y$.

3. If $f(x) = x^2$, and if someone tells you that f takes the value 4, you *don't* know exactly where: x could equal 2 or -2. However, *if* you restrict yourself to looking only for positive values of x, then if f takes value y, you know that the original x is $\sqrt{y}$.

Below is a visual way to think about this process. Drawn is the graph of f. Given a value y, can you figure out which value of x satisfies the equation $f(x) = y$? If so, that means that the coordinate x now becomes a function of y—you input y, and you output x—and we can call this function g.



Warning 6.3.2. While we were diligent about drawing g as a function of y before, from now on, we must now be comfortable realizing that letters are just letters, and we don't care if g takes inputs to be symbols that look like "x," or symbols that look like "y"; that is, g will often be treated as a function of x, too.

Definition 6.3.3. Let f be a function. We say that a function g is a *left inverse* to f if

$$(g \circ f)(x) = x.$$

Put another way, g "remembers" the original value x that outputted $f(x)$.

We also say that f is a *right inverse* to g. Put another way, f "knows" that if $g(\text{something}) = x$, then something $= f(x)$.

6.4 Derivatives of right inverses

It turns out that if we know the derivatives of a function g, then—if g has a right inverse f—we can figure out the derivatives of the right inverse f.

Lemma 6.4.1. Let g be a function, and suppose that f is a right inverse to g, defined on some open interval containing x. Suppose also that g is differentiable at $f(x)$, and that $g'(f(x)) \neq 0$. Then

$$f'(x) = \frac{1}{g'(f(x))}. \tag{6.4.1}$$

That is, the derivative of f at x is computed by dividing 1 by the derivative of g *at* $f(x)$.

Proof. Let's look at the following string of equalities:

$$\begin{align} 1 &= (x)' \tag{6.4.2} \\ &= (g \circ f)'. \tag{6.4.3} \end{align}$$

The first equality is our knowledge of the derivative of the function x. The next equality is using the hypothesis that g is a right inverse to f, so that $f \circ g = x$.

In total, what this string of equalities says is that the function on the righthand side is equal to the (constant!) function on the lefthand side. So let's evaluate at some point x. We have

$$1 = (g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

We can divide both sides by $g'(f(x))$ so long as this number isn't zero; so we find:

$$\frac{1}{g'(f(x))} = f'(x) \qquad \text{when } g'(f(x)) \neq 0.$$

This is what we wanted. ☐

Example 6.4.2. $f = \ln(x)$ is a right inverse to $g(x) = e^x$. This is because

$$g \circ f(x) = e^{\ln x} = x.$$

We know the derivative of $g(x) = e^x$, so we can use the lemma to find the derivative of $f(x) = \ln(x)$! Let's try:

$$\begin{aligned}
(\ln(x))' &= f'(x) && (6.4.4)\\
&= \frac{1}{g'(f(x))} && (6.4.5)\\
&= \frac{1}{e^{f(x)}} && (6.4.6)\\
&= \frac{1}{e^{\ln(x)}} && (6.4.7)\\
&= \frac{1}{x}. && (6.4.8)
\end{aligned}$$

. The first equality is by definition of f. The next equality is using Lemma 6.4.1. The rest is just plugging in our knowledge of g' and ln.

6.5 The derivative of natural log

The example from the last page is important, so let's record this as a theorem. (You will be expected to know this:)

Theorem 6.5.1 (The derivative of ln)**.** The derivative of ln is "one over x." That is,

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Exercise 6.5.2. Find the derivative of the following functions:

(a) $\ln(7x)$

(b) $\ln(7x - 9)$

(c) $\ln(\cos(x))$

(d) $\ln(\sin(x))$

(e) $\ln(x^2 - 10)$

(f) $\ln(x^2e^x)$. (For this, you may want to remember that $|ln(a+b) = \ln(a) + \ln(b)$.)

Possible solutions. We will use the chain rule for all of these.

(a) $(\ln(7x))' = \frac{1}{x}$. This is by writing $\ln(7x)$ as a composition of $f(x) = 7x$ and $g(x) = \ln(x)$. Alternatively, you could remember that $\ln(ab) = \ln(a) + \ln(b)$. So $\ln(7x) = \ln(7) + \ln(x)$. Then $(\ln(7) + \ln(x))' = 0 + (\ln(x))' = 1/x$.

(b) $(\ln(7x-9))' = \frac{7}{7x-9}$. This is by writing $\ln(7x)$ as a composition of $f(x) = 7x - 9$ and $g(x) = \ln(x)$.

(c) $\ln(\cos(x))' = -\tan(x)$, the tangent of x. To see this, write $\ln(\cos(x))$ as a composition of $f(x) = \cos(x)$ and $g(x) = \ln(x)$. Then by the chain rule, we have

$$\ln(\cos(x))' = \frac{1}{\cos(x)} \times (\cos(x))' = \frac{1}{\cos(x)} \times (-\sin(x)) = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

(d) $\ln(\sin(x))' = \cot(x)$ otherwise known as cotangent of x. To see this, write $\ln(\sin(x))$ as a composition of $f(x) = \sin(x)$ and $g(x) = \ln(x)$. Then by the chain rule, we have

$$\ln(\sin(x))' = \frac{1}{\sin(x)} \times (\sin(x))' = \frac{1}{\sin(x)} \times \cos(x) = \frac{\cos(x)}{\sin(x)} = \cot(x).$$

(e) $(\ln(x^2-10))' = \frac{2x}{x^2-10}$.

(f) $\ln(x^2e^x) = \ln(x^2) + \ln(e^x) = \ln(x) + \ln(x) + x = 2\ln(x) + x$. Hence

$$(\ln(x^2e^x))' = (2\ln(x) + x)' = \frac{2}{x} + 1.$$

Here is another way to do this:

$$(\ln(x^2e^x))' = (\ln(x^2) + \ln(e^x))' = 2x(\frac{1}{x^2}) + (x)' = \frac{2}{x} + 1.$$

$\square$

6.6 Bonus: Derivatives of other inverse functions. And, why e?

6.6.1 Inverses, revisited

We learned that

$$\text{If } f(x) = e^x, \text{ then } f'(x) = e^x.$$

That is, e^x is some seemingly special function—it is its own derivative!

Based on this fact (which we took for granted), we learned about *inverses*, and learned that we can try to compute derivatives of inverse functions. As an example, we recalled that $\ln x$ is an inverse to e^x, and we deduced that

$$\text{If } f(x) = \ln x, \text{ then } f'(x) = \frac{1}{x}.$$

But let's talk a little bit about what an inverse function is. I am going to ignore the words "right" and "left" for today, to simplify things.

Informally, an *inverse to* f is a function that "undoes" f. For example, f takes a number x, and outputs a number called $f(x)$. What does it mean to undo this? Well, to undo this process would be to take a number called $f(x)$, and output/return a number called x.

Example 6.6.1. If $f(x) = e^x$, f takes a number, then outputs e to that number. For example, f takes a number like 2, and outputs a number e^2, which is roughly $7.38905609893\ldots$.

If there is to be a function g that applies *undo* to f, it must take the number $7.38905609893\ldots$ and output 2. More accurately, if g sees an input called e^2, it ought to return 2. And more generally, if g sees an output balled e^{blah}, g should output blah.

The great thing is that you had already seen such a function in precalculus—this function is called ln, or the natural log.

6.6.2 arcsin as an inverse to sin

You have seen other examples of inverses. For example, sin is a function that takes in an *angle*, and outputs a *height* (of a point on the unit circle). Do you think we could go backward? For example, if we are given a *height* of a point on the unit circle, we might be able to say what angle that point is at.

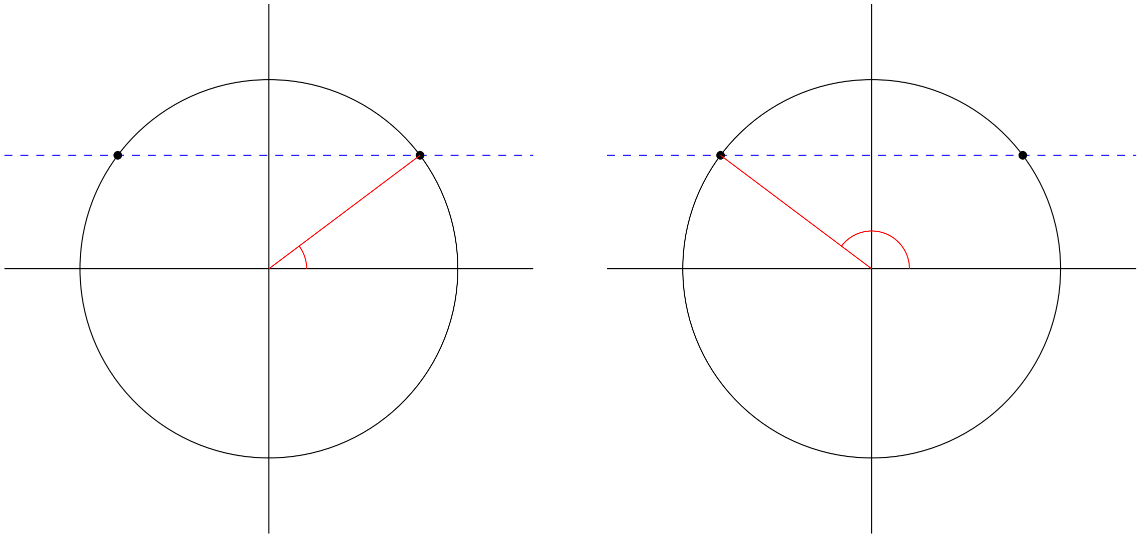


Figure 6.1: A single height (the blue dashed line) determines two possible points (the black dots) on the circle, hence two possible angles (in red).

Above is a picture of a blue dashed line (drawn to indicate, for example, a line of height 0.6). We see an immediate issue, which is that the blue dashed line (i.e., a height) actually defines *two* possible points on the circle. So it's not clear which angle we should take. See Figure 6.2.

So let's just *agree* as a community that, if we want to specify a point or an angle from a height, we will always take the point or angle on the *right half* of the unit circle.

Definition 6.6.2. We will call this angle the *arcsine*, or *inverse sine*, of the height. So, given a height x, we let

$$\arcsin(x)$$

denote the angle formed by the point (on the right half of the unit circle) with height x.

Remark 6.6.3. sin is a function that takes an angle and returns a height. arcsin does "the reverse," by taking a height and returning an angle. By design, is it an inverse to sin (along the right half of the unit circle—that is, for angles between $-\pi/2$ and $\pi/2$).

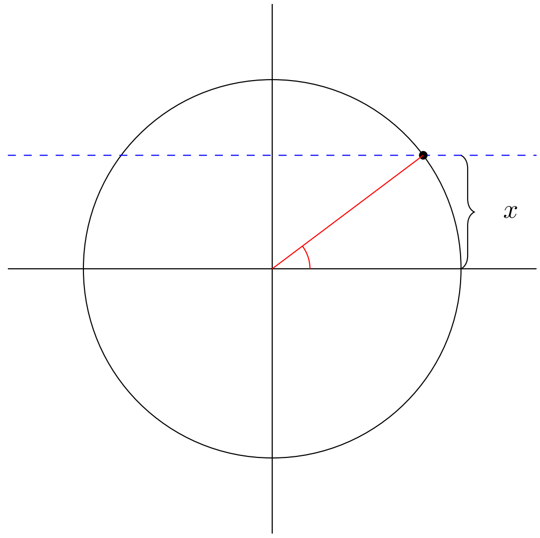


Figure 6.2: The red angle is $\arcsin(x)$ (in radians).

6.6.3 The derivative of arcsin

As we saw last time, when we have an inverse function to f, we can find the derivative of the inverse function in terms of the derivative of f. Let's recall the work. Suppose we have a function $g(x)$ so that $f(g(x)) = x$. Then

$$\begin{aligned} 1 = (x)' &= (f(g(x))' \\ &= f'(g(x)) \cdot g'(x) \end{aligned} \tag{6.6.1}$$

so we know that

$$g'(x) = \frac{1}{f'(g(x))}.$$

If we want to find the derivative of $\arcsin(x)$, we set $g(x) = \arcsin(x)$ and $f(x) = \sin(x)$. So the above formula becomes

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}. \tag{6.6.2}$$

So we have an expression for the derivative of $\arcsin(x)$; but we'd like things to be more transparent—we'd like to make the righthand side more understandable.

So let's think carefully. $\arcsin(x)$ is the angle (on the right half of the unit circle) formed by a point of height x. And $\cos(\arcsin(x))$ is the horizontal coordinate formed by a point at that angle. In other words, $\cos(\arcsin(x))$ is the horizontal coordinate of the black dot in Figure 6.2.

At this point, we can find what $\cos(\arcsin(x))$ is in terms of x! We use the Pythagorean theorem. On any circle centered at the origin, and any point on that circle, we know

$$(\text{radius})^2 = (\text{horizontal coordinate})^2 + (\text{vertical coordinate})^2.$$

We are on the unit circle, so our radius is 1. We also have names already for our vertical and horizontal coordinates:

$$(1)^2 = (\cos(\arcsin(x)))^2 + x^2.$$

Doing some algebra, we find

$$\cos(\arcsin(x)) = \sqrt{1 - x^2}. \tag{6.6.3}$$

Note that we have used the *positive* square root of $1-x^2$, because we agreed that our point is on the right half of the circle. (On the left half of the circle, our horizontal coordinate would be negative.

To summarize, by combining (6.6.2) and (6.6.3), we find the following:

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}$$

Some professors may expect you to know the derivative of arcsin. This is a derivative that you may well forget in a week after the course ends. On the other hand, trying to figure out the derivative on your own (by going through the process we just completed) can take a long time on a test. So, for the purposes of this course, I would recommend that you just memorize the formula. Future classes, especially Calculus II, will likely expect you to be able to recall this derivative from memory.

However, for the purposes of your future and for your understanding, you should understand that arcsin is just the inverse to sin, so the chain rule allows you to compute its derivative in terms of the derivative of sin. Then, some geometry (using the Pythagorean theorem) allows you to compute the derivative of arcsin without using any sines or cosines in the final answer.

6.7 Bonus: What's up with e?

I don't know how you were introduced to the number e, but let's talk about a really cool reason to care about e.

First, let's consider the following functions:

1. $f(x) = 2^x$
2. $f(x) = e^x$
3. $f(x) = 3^x$
4. $f(x) = 5^x$.

You know how to take the derivatives of these functions. For example, to take the derivative of 2^x, you might write

$$2^x = e^{(\ln 2)\cdot x}$$

so

$$(2^x) = \ln 2 e^{(\ln 2)\cdot x} = \ln 2 \cdot 2^x.$$

In other words, when $f(x) = 2^x$, we see that

$$f'(x) = \ln 2 \cdot 2^x = \ln 2 \cdot f(x).$$

Taking the derivatives of the other functions, we see

1. $f(x) = 2^x \implies f'(x) = \ln 2 f(x)$
2. $f(x) = e^x \implies f'(x) = f(x)$
3. $f(x) = 3^x \implies f'(x) = 3f(x)$
4. $f(x) = 5^x \implies f'(x) = 5f(x)$.

So e is quite a special number! In fact, it's the *only* number such a that the derivative of a^x is equal to a^x itself.

That's what's so "natural" about e, and why we call ln, or log base e, the "natural log."[1]

[1]By the way, you might have wondered why "natural log" is written ln as opposed to *nl*. Well, ln comes from the French, *logarithme naturel*, which you might guess means natural logarithm. But just like in Spanish, the order of the adjective and noun are flipped. (In Spanish, it's *logaritmo natural*.) Hence the ln, as opposed to nl.

6.8 Why e^x is its own derivative

I have claimed that the derivative of e^x is itself. How might we see that?

First, let $f(x) = a^x$, where a is some number. (It could be 2 or 3, but let's ignore what number it is exactly so that we can see a pattern.)

Then as usual, the derivative of f is computed by taking the limit of the difference quotient:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

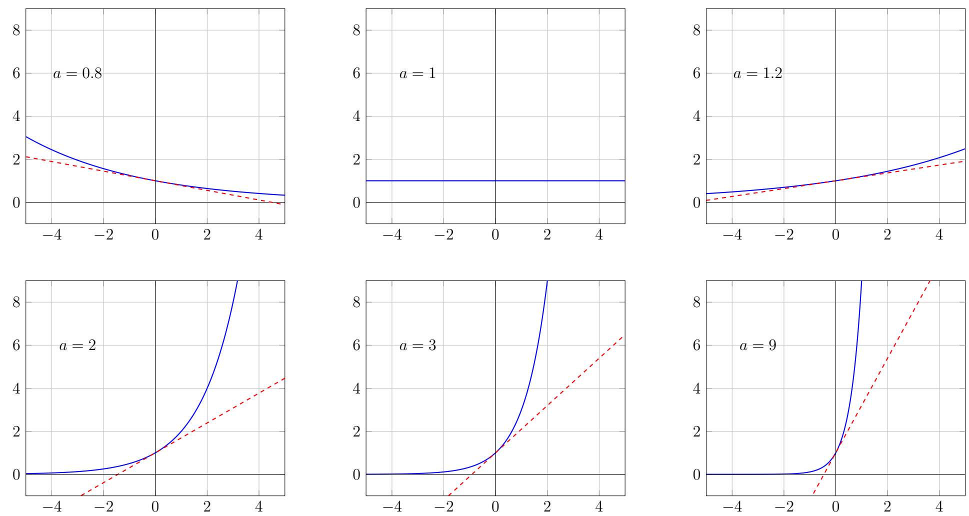
Plugging what f is, we find

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} a^x \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h a^0 - a^0}{h} \\
&= a^x \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} \\
&= a^x \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\
&= a^x f'(0).
\end{aligned} \tag{6.8.1}$$

In other words, the derivative of a^x is always given by *the value of a^x times the derivative of a^x at zero.*

In other words, the derivative of a^x is pretty much the same thing as a^x, but scaled by whatever the derivative at $x = 0$ is.

We can draw the graphs of $f(x) = a^x$ for different values of a:



The tangent line at $x = 0$, for each value of a, is drawn. Note that the tangent line has negative slope at $a = 0.8$ (when $a < 1$), is flat—and hence has slope zero—when $a = 1$, then the slope keeps getting positive, and bigger and bigger, as a increases.

Thus, for *some* value of a, the slope must equal exactly 1!

And why does that matter? Well, for that value of a, we thus have that $f'(0) = 1$. Hence for that value of a, $f'(x) = f(x)$.

You can *define* e to be the value of a for which the limit of $(a^h - 1)/h$ as $h \to 0$ is given by 1. This is probably the craziest way you've ever seen a number defined, and it really takes a very clever person to think up of the *existence* of such a number without constructing it. But indeed, we have done this as a civilization, and we can now utilize it.

6.9 For next time

You should be comfortable finding derivatives of functions involves e^x and ln. For example, you should be able to find f' for each of the following functions f:

(a) $f(x) = e^x$

(b) $f(x) = e^{3x}$

(c) $f(x) = e^{3x+2}$

(d) $f(x) = 3e^x$

(e) $f(x) = 5^x$

(f) $f(x) = 5^{3x}$

(g) $f(x) = \ln(x)$

(h) $f(x) = \ln(3x)$

(i) $f(x) = \ln(x+3)$

(j) $e^{\sin(x)}$

(k) e^{x^2}

(l) $\ln(\sin(x) + \cos(x))$

(m) $\ln(\sin(x))$

(n) $\ln(x^3 - x)$