Lecture 5

The chain rule; review of exponentials and logarithms

Here are the summaries of the rules/laws we know for derivatives so far:

Rule/Law	For derivatives
Constants	$(C)' = 0.$
Scaling	$(mf)' = mf'$
Sums	$(f' + g') = f' + g'$
Powers	$(x^n)' = nx^{n-1}.$
Composition	$(f \circ g)' =???$
Products	$(fg)' =???$
Quotients	$(f/g)' =???$

Today, we are going to practice taking derivatives of *compositions*. The rule we use to compute derivatives of composition is called the *chain rule*. Then, we'll review some exponentiation and logarithmic computations in preparation for next time.

5.1 Review of compositions

The hardest part of applying the chain rule, for most calculus students, is actually understanding what a composition of functions is.

Remember that functions take inputs and produce outputs. A *composition* happens when a second function uses a first function's output as the second function's input. If you like, a composition is like a relay race in track and field. The first function passes a *number* onto the second function (instead of a baton).

When f is a function, and g is another function, we write

$$g \circ f$$

for the composition. This is a new function that, by definition, we evaluate at a number x by the following method:

$$(g \circ f)(x) = g(f(x)).$$

The righthand side, in words, says: Apply f to x, and whatever $f(x)$ is, plug it into g.

Example 5.1.1. Let $f(x) = x + 2$ and $g(x) = x^2$. Then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= g(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 2. \end{aligned} \tag{5.1.1}$$

Example 5.1.2. Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= g(\sin(x)\cos(x)) \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

If you like, this last expression could also be written as

$$\sin(x)^2\cos(x)^2 + 3\sin(x)\cos(x) + 2 \qquad \text{or} \qquad \sin^2(x)\cos^2(x) + 3\sin(x)\cos(x) + 2$$

You can also try to compute the "outside function" first.

Example 5.1.3. Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= (f(x))^2 + 3f(x) + 2 \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

5.2 Dependencies multiply

"Composite" function seems like an abstract concept. The everyday way to think about a composition—for calculus purposes—is that we have an outside function that depends on another value. In daily life, this manifests in the following way: Some quantity A depends on quantity B, and quantity B depends on quantity C; so ultimately, quantity A depends on quantity C. Here are some examples:

Example 5.2.1. Let's suppose that a rocket is ascending upward at 1,000 kilometers an hour. Suppose that air pressure changes by -12 kiloPascals every kilometer. Then we can figure out the rate at which air pressure is changing:

$$\frac{1,000\text{km}}{\text{hr}} \cdot \frac{-12\text{kPa}}{\text{km}} = \frac{1,000 \cdot -12\text{kPa}}{\text{hr}} = \frac{-12,000\text{kPa}}{\text{hr}}.$$

That is, air pressure is changing for the rocket at -12,000 kiloPascals per hour. Note that we are doing "unit cancellation," sometimes known as "dimensional analysis," to get rid of the "km" units. (They "cancel out," if you like.)

In this example, the height of the rocket depends on time, while air pressure depends on the height. So ultimately, air pressure depends on time—and this dependency can be computed explicitly by *multiplying* the dependencies (as opposed to adding, for example).

Example 5.2.2. Saxon's car has a fuel efficiency of 35 miles per gallon. Saxon is driving the I-35 at 70 miles per hour. At what rate is he using up his fuel?

$$\frac{70\text{miles}}{\text{hour}} \times \frac{1\text{gallon}}{35\text{miles}} = \frac{2\text{gallons}}{\text{hour}}.$$

In other words, Saxon is using 2 gallons per hour. (Note: Most cars with a particular fuel efficiency most likely perform under that fuel efficiency if they are driven 70 miles per hour.)

Note that I knew that a fuel efficiency of 35 miles per gallon can be represented as $\frac{1}{35}$ gallons per mile – this isn't a fact about vehicle engineering, it's a general principle about how to think about dependencies. Make sure to think this through if it's not making sense.

Example 5.2.3. Likewise, if a scuba diver is ascending toward the surface, their depth changes with time, and the water pressure changes with depth. So ultimately, the water pressure a scuba diver experiences varies with time.

The rate at which the diver ascends is incredibly important. This is because if the diver ascends too quickly, the water pressure changes very quickly; it turns out that this “un-dissolves” some gases in the diver’s bloodstream too quickly, and can result in a condition called the bends. For a diver, this is a potentially deadly experience.

Example 5.2.4. The energy production rate of a wind turbine depends on how quickly the turbine is spinning, and how quickly the turbine spins depends on the wind speed. We can model energy production rate E as a function of spin rate S, and dE/dS represents the rate of change of E with respect to S. On the other hand, S is a function of wind speed W, and dS/dW measures the rate of change of S with respect to W. Then (roughly speaking) we expect $dE/dS \cdot dS/dW$ to compute dE/dW.

5.3 The Chain Rule

Theorem 5.3.1 (Chain rule)**.** Suppose that g is differentiable at x, and that f is differentiable at $g(x)$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Put another way,

$$\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$$

I want to emphasize in words what the chain rule says: If you want to compute the derivative of $f \circ g$ at x, then you must compute two things:

1. The derivative of f **at g(x)**, and

2. The derivative of g at x.

The product of these two numbers gives the derivative of $f \circ g$ at x. Remember, we’ve seen that dependencies multiply. And that’s what the chain rule encapsulates – if you multiply the way the outside function depends on its input, with the way the inside function depends on its input, you are calculating the way in which the composite function depends on its input.

Example 5.3.2. The function $\cos(x)^4$ is a composite of two functions: First, take the cosine of x. Then, raise the output to the 4th power. Using the notation from the Chain rule above, we have that $f(x) = x^4$ and $g(x) = \cos(x)$. Then

$$((\cos(x))^4)' = f'(g(x)) \cdot g'(x) = 4(g(x))^3 \cdot (-\sin(x)) = -4\cos(x)^3 \sin(x).$$

Exercise 5.3.3. Using the chain rule, find the derivative of the following functions:

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

5.4 Identifying compositions

For many calculus students, the hardest part about taking derivatives is knowing whether we can use the chain rule in a particular situation. Why is this so hard? Well, in past classes, you've learned how to compose two functions, but you haven't learned to recognize whether a given function *arises* as a composition. Moreover, to use the chain rule, you need to be able to recognize the functions that are being composed.

Example 5.4.1. Let's write each of the functions below as a composition $g \circ f$. Importantly, let's identify the functions f and g.

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

Solution:

(a) What this expression tells us to do is to *first* evaluate $\sin(x)$, and then cube the result. So the first function is $f(x) = \sin(x)$ and the second, or "outside" function is $g(x) = x^3$.

(b) This expression tells us to first cube x, and then take sin of the result. So the first function is $f(x) = x^3$, and the second, or "outside" function is $g(x) = \sin(x)$.

(c) This expression tells us to take a number x, and first evaluate $x^4 + 3x^3 - 2$, and then take cos of the result. So $f(x) = x^4 + 3x^3 - 2$, and $g(x) = \cos(x)$.

You should check in all three examples that $(g \circ f)(x)$ indeed gives rise to the original expression.

5.5 Applying the chain rule

Now let's apply the chain rule.

Example 5.5.1. Find the derivative of $\sin(x^2+5)$.

Solution. We must first recognize that we behold a composition of two functions: On the outside is sin, while the inside is x^2+5. Hence we can use the chain rule.

$$\frac{d}{dx}(\sin(x^2+5)) = (\frac{d}{dx}\sin)(x^2+5) \cdot \frac{d}{dx}(x^2+5).$$

This is a product of two factors: The first factor, on the left, is the derivative of sin, evaluated at x^2+5. The second factor, on the right, is the derivative of x^2+5 (evaluated at x).

Because we know $\frac{d}{dx}\sin = cos$, and that $\frac{d}{dx}(x^2+5) = 2x$, we conclude:

$$\frac{d}{dx}(\sin(x^2+5)) = \cos(x^2+5) \cdot 2x.$$

Or, in more palatable notation,

$$\frac{d}{dx}(\sin(x^2+5)) = 2x\cos(x^2+5).$$

Exercise 5.5.2. Find the derivatives of the following functions:

1. $(\cos(x)+\sin(x))^3$
2. $\cos(\sin(x))$
3. $\cos(2x^4)$.

Exercise 5.5.3. We do not yet know how to take derivatives of a function like $h(x) = x^{1/3}$. *However*, we do know that if $g(x) = x^3$, then $g(h(x)) = x$.

Using this, and the chain rule, can you find a formula for $h'(x)$? That is, can you compute the derivative of $x^{1/3}$?

5.6 Exponentials, logarithms, and e (A primer)

Remark 5.6.1. If you are already comfortable with functions like e^x and $\ln x$, and how they relate to functions like 2^x and $\log_2 x$, you can focus on Section **??**.

From today onward, we'll need to be prepared to use exponentials and logarithms.

Consider the function $f(x) = 4^x$. You have seen this in precalculus. In fact, you probably knew that

$$4^0 = 1, \qquad 4^1 = 4, \qquad 4^2 = 4 \times 4 = 16, \qquad 4^3 = 4 \times 4 \times 4 = 64,$$

et cetera, back in high school. The cool fact is that even if x is not an integer, 4^x is a number that makes sense.

Example 5.6.2. Here are some examples:

1. It makes sense to raise something to a negative power:

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

   More generally, we have that

$$4^{-n} = \frac{1}{4^n}.$$

2. It makes sense to raise something to a fraction:

$$4^{\frac{1}{3}} = \sqrt[3]{4} \text{ is the cube root of 4.}$$

   More generally, we have that

$$4^{1/n} = \sqrt[n]{4}$$

   is the nth root of 4. This root is the unique positive number so that its nth power is equal to 4.

3. Another fraction example:

$$4^{\frac{2}{3}} = \sqrt[3]{4^2}.$$

   (Note that this also equals $(\sqrt[3]{4})^2$.) Put into English, this means that $4^{2/3}$ is the cube root of 4^2, or the square of the cube root of 4. More generally,

$$4^{\frac{a}{b}} = \sqrt[b]{4^a} = (\sqrt[b]{4})^a.$$

5.6.1 Exponent laws!

Let's see why the above statements are true.

You may have learned about the exponent laws in precalculus or back in high school. One of these laws says:

$$4^{a+b} = 4^a \cdot 4^b$$

That is, exponentiation takes "addition" to "multiplication." For example, $4^7 = 4^{2+5} = 4^2 \cdot 4^5$. Another law says:

$$4^{a \cdot b} = (4^a)^b = (4^b)^a.$$

This means that exponentiation takes "multiplication" to "powers." For example, $4^{21} = 4^{3 \cdot 7} = (4^3)^7$. Also, we have that $4^{21} = (4^7)^3$.

Example 5.6.3. Let's verify that the exponent laws are consistent with our knowledge of math. We have:

$$4^{2+3} = 4^5 = 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4 \times 4) = 4^2 \times 4^3.$$

So indeed, $4^{2+3} = 4^2 \cdot 4^3$.

We also have:

$$4^{2 \cdot 3} = 4^6 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4) \times (4 \times 4) = (4 \times 4)^3 = (4^2)^3.$$

Remark 5.6.4 (Reminder)**.** Let me also remind you that *anything* to the 0th power is equal to 1. For example, $5^0 = 1$. Likewise, $\pi^0 = 1$.

And, anything to the 1st power is that anything again. For example, $5^1 = 5$.

Remark 5.6.5 (The reasoning for fractional and negative powers)**.** Knowing these laws is how you *create* the definitions for things like 4^{-3} and $4^{1/5}$. Indeed, if you know what 4^3 is, and if you desire the law $4^{3+(-3)} = 4^3 \cdot 4^{-3}$ to be true, you *must* conclude that 4^{-3} is equal to $1/4^3$. For example,

$$4^3 \cdot 4^{-3} = 4^{3+(-3)} = 4^0 = 1.$$

Dividing both sides by 4^3, we see

$$4^{-3} = \frac{1}{4^3}.$$

Likewise, the other law of exponent tells us

$$5 = 5^1 = 5^{\frac{1}{2} \cdot 2} = (5^{\frac{1}{2}})^2.$$

Taking the square root of both sides, we find

$$\sqrt{5} = 5^{\frac{1}{2}}.$$

There's nothing special about the number 5 here; anything to the $\frac{1}{2}$ power is the square root of that anything. Likewise, anything to the $\frac{1}{3}$ power is the cube root.

5.6.2 The number e

The number e is called Euler's constant sometimes, but it's usually just called e. (Eeee!) In your previous math classes, you probably didn't have too much reason to care about this deeply, except that it has some interesting roots in banking. However, you will see why e is important in calculus.

For now, let me just say that e is an irrational number, and here are the first few digits of its decimal expantion:

$$2.718281828459045235360287471352\ldots \tag{5.6.1}$$

We will soon be dealing with the function $f(x) = e^x$. You should think of this function as behaving very much like $f(x) = 4^x$. For example, we have that

$$e^0 = 1, \qquad e^1 = e, \qquad e^2 = e \times e \approx 7.38905609\ldots.$$

(We compute e^2 using a computer or calculator; if we have a lot of time at the end of this course, we'll see *how* a computer does this!)

5.6.3 The logarithm

The logarithm base n of a number x is written

$$\log_n x.$$

The number $\log_n x$ is the number you need to raise n to in order to obtain a value x. For example,

$$\log_3 9 = 2.$$

This is because 2 is the number such that $3^2 = 9$. As another example,

$$\log_3 81 = 4.$$

(Just try computing 3^4 to see why this is true.)

Put another way, it is always true that

$$3^{\log_3 x} = x.$$

We say that the logarithm base n is the *inverse* function to exponentiation base n. (Put another way, if the output of the logarithm becomes the input of the exponential, the final output is the initial input.)

In fact, it is also true that

$$\log_3(3^x) = x.$$

Exercise 5.6.6. You should be able to compute the following:

(a) $\log_2 8$

(b) $\log_3 243$

(c) $\log_2 \sqrt{2}$

(d) $\log_\pi \pi^3$.

5.6.4 Natural logarithm

Because e is so special[1], we give a special name to the logarithm base e. We define the *natural logarithm*, or the *natural log*, to be the logarithm base e, and we denote it as follows:

$$\ln$$

So for example,

$$\ln e = 1, \qquad \ln(e^3) = 3.$$

In general ln of a nice integer looks crazy; for example,

$$\ln 2 = 0.69314718056\ldots$$

so if you like integers and rational numbers, ln is not your best friend. But it will become a better friend as we realize how important ln and e are in calculus—in fact, it is probably one of the most convincing pieces of evidence that crazy, *transcendental numbers* like e have a place in our mathematical universe.[2]

In calculus, it will be very useful to know how to convert expressions like

$$5^{\text{some power}}$$

into expressions base e; that is, into expressions like

$$e^{\text{some other power}}.$$

[1] We will see why in the coming lectures

[2] There is another transcendental number, π, that is obviously very important to mathematics. If you don't know what a transcendental number is, don't worry; they are a special kind of irrational number.

Example 5.6.7. Let us convert 5^3 into an exponent with base e. Here is our work:

$$5^3 = (e^{\ln 5})^3 \tag{5.6.2}$$
$$= e^{(\ln 5)\cdot 3} \tag{5.6.3}$$
$$= e^{3 \ln 5}. \tag{5.6.4}$$

The first equality is using the definition of logarithm base e. (Note that we don't need to know how to calculate $\ln 5$, but we know that it exists as a number, so we just use it.) The next equality follows from an exponent law: Exponentiation exchanges multiplication of powers with iterated powers. The last line is just re-writing the same expression in a nicer way.

5.6.5 Exponentials for non-rational powers

Now, you may not have thought deeply about how to calculate something like 4^x when x is, say, an irrational number. In this class, you only need to know this *can* be done, and not *how* to do it.

So you only need to read this section if you're curious about how something like 4^π is computed. I'll illustrate by example.

Example 5.6.8. For example, how would you compute 4^π? It's a three-step process.

First, we choose a collection of numbers that approximates π really well. For example, we could choose

$$3, \qquad 3.1, \qquad 3.14, \qquad 3.141, \qquad 3.1415, \qquad 3.14159, \qquad \ldots$$

and so forth. Note that each of these numbers can be written as a fraction. (For example, 3.14 is equal to 314/100; you can simplify this fraction if you like.) In particular, we *know* how to calculate each of the following numbers:

$$4^3, \qquad 4^{3.1}, \qquad 4^{3.14}, \qquad 4^{3.141}, \qquad 4^{3.1415}, \qquad 4^{3.14159}, \qquad \ldots$$

and so forth.

Calculating all these numbers is the second step. For your edification, here are the answers:

$$64, \qquad 73.516\ldots, \qquad 77.708\ldots, \qquad 77.816\ldots, \qquad 77.870\ldots, \qquad 77.880\ldots,$$

and so forth.

Now, here is the third and most fun/difficult step. We have to *prove* that this collection of numbers "converges" to some number—put another way, that this sequence has a limit.[3] Then we *define* 4^π to be this limit.

Exercise 5.6.9. Simplify the following expressions:

(a) $e^{\ln 3}$

(b) $e^{\ln e}$

(c) $e^{\ln x}$

(d) $e^{\ln 1}$

(e) $e^{\ln \pi}$

(f) $(e^3)^{1/3}$

(g) $(e^3)^{2/3}$

(h) $(e^3)^{4/3}$

(i) $(e^x)^2$

(j) $(e^x)^7$

(k) $\ln(e^3)$

(l) $\ln(e^\pi)$

(m) $\ln(e^x)$

(n) $\ln(e^3 \cdot e^5)$

(o) $\ln(e^3 \cdot e^x)$

(p) $\ln(e^3 \cdot e^{-3})$

(q) Convert $5 = e^\heartsuit$. Find $\heartsuit$, using ln in your answer.

(r) Convert $5^{3x} = e^\heartsuit$. Find $\heartsuit$, using ln in your answer.

(s) Convert $e^{3x} = 5^\heartsuit$. Find $\heartsuit$, using ln in your answer.

(t) Convert $e^{9x} = 5^\heartsuit$. Find $\heartsuit$, using ln in your answer.

[3]Warning: This notion of limit is slightly different from the notion of limit we have discussed before. This is the limit of a *sequence of numbers*, while we have discussed in this class the limit of a *function* at a point.

5.7 For next time

For next time, you should be able to take derivatives of functions like those appearing in Exercise 5.3.3. You should also be comfortable with all the problems in Exercise 5.6.9.