Lecture 4

Drawing derivatives; Derivatives of sine and cosine

Last time we learned how to take derivatives of polynomials. This was a very precise calculation – very mathy!

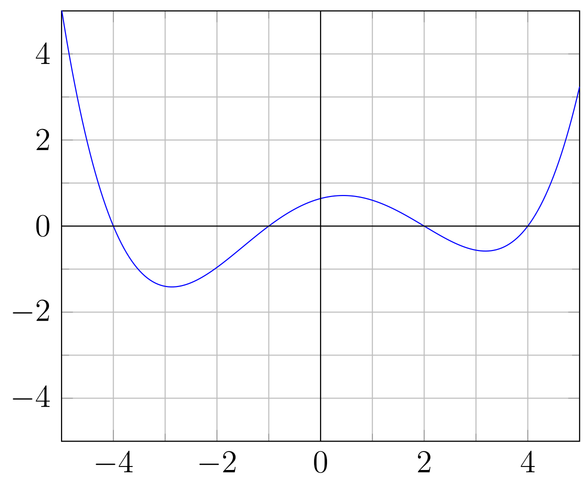
Let's now think about some qualitative tools – how to draw derivatives. Then, we'll learn how to take derivatives of $\sin(x)$ and $\cos(x)$.

4.1 Drawing derivatives

Let f be a function. Last time we saw that the derivative f' is a *new* function. Because every function has a graph, we can now ask what the graph of f' looks like.

Let's start with some warm-up exercises.

Exercise 4.1.1. Below is the graph of a function $f(x)$.

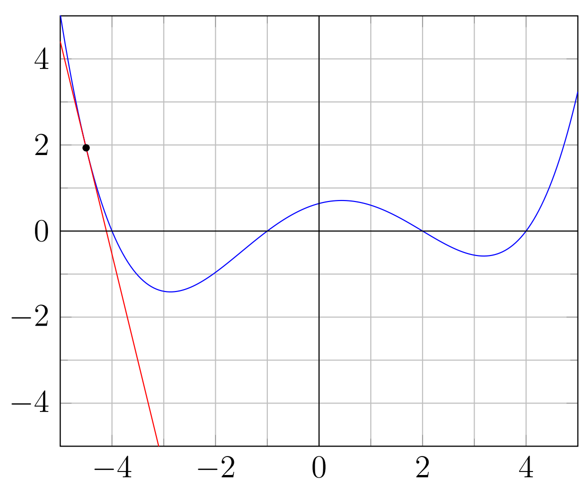


(a) Draw the tangent line to the graph of f at the point $x = -4.5$. Is the slope positive or negative or zero?

(b) Draw the tangent line to the graph of f at the point $x = -2.9$. Is the slope positive or negative or zero?

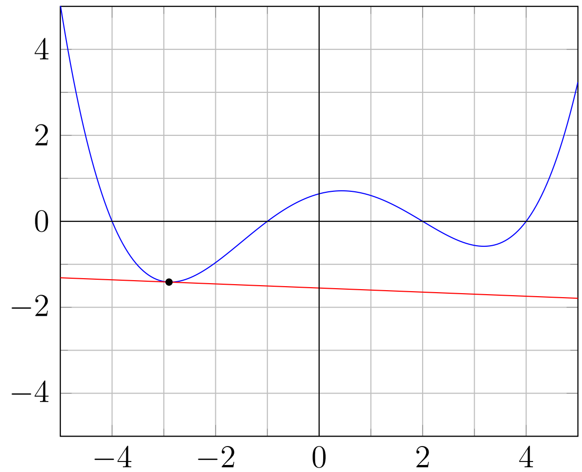
(c) Draw the tangent line to the graph of f at the point $x = -1$. Is the slope positive or negative or zero?

Some possible solutions. (a) Here is the tangent line to the graph of f where $x = -4.5$. The black dot indicates the point on the graph where $x = -4.5$.



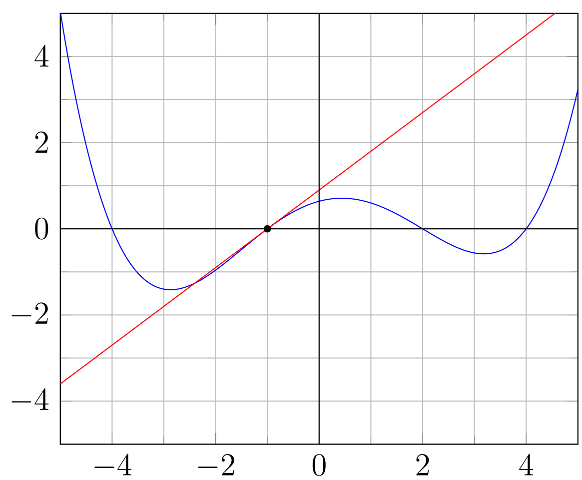
The slope is negative, because the line moves downward as we move from left to right.

(b) Here is the tangent line to the graph of f where $x = -2.9$. The black dot indicates the point on the graph where $x = -2.9$.



The slope is negative, because the line moves downward as we move from left to right. However, many people might have drawn a horizontal tangent line – that's completely acceptable for this problem, because this problem was about "eyeballing" or estimating the tangent line. (Without being given a formula for f, it's impossible to know the correct answer for sure.)

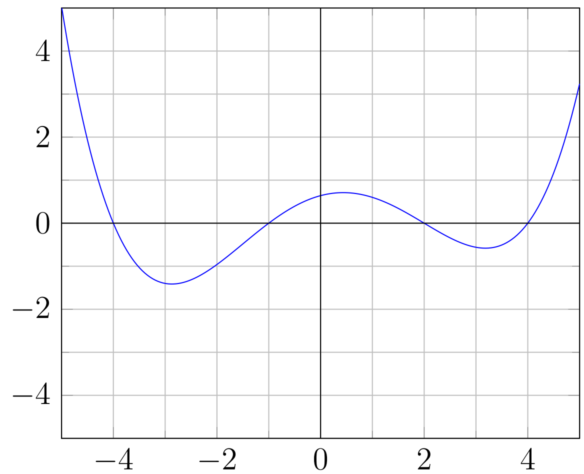
(c) Here is the tangent line to the graph of f where $x = -1$. The black dot indicates the point on the graph where $x = -1$.



The slope is positive, because the line moves upward as we move from left to right.

□

Exercise 4.1.2. Below is the graph of the same function $f(x)$ from before.



(a) Estimate using your eyes: At what values of x is the derivative of f equal to zero?

(b) Estimate using your eyes: At what values of x is the derivative of f positive?

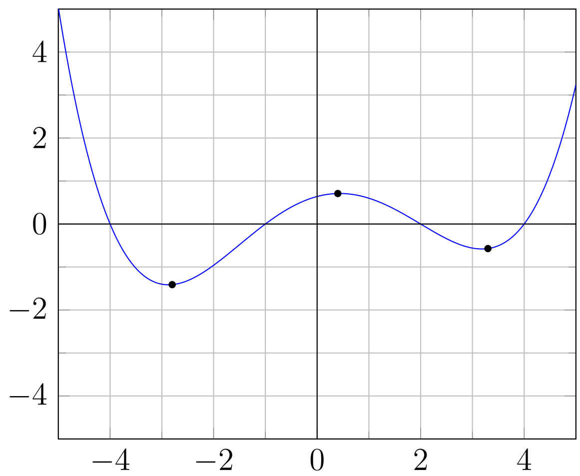
(c) Estimate using your eyes: At what values of x is the derivative of f negative?

Some solutions. (a) Let's first tackle the first part of the example. When is the derivative of a function equal to zero? Here, because the problem itself only gives us a graph (something geometric) and no actual expression or algebraic formula for f, the only tool we have is our geometric intuition/knowledge about the derivative.

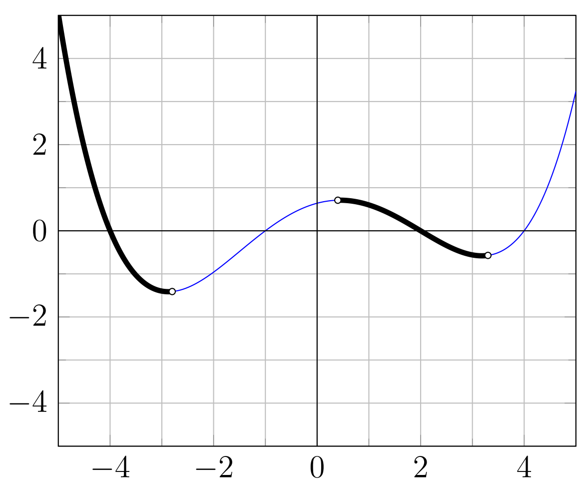
Remember that the derivative measures the slope of something – the slope of the tangent line. So, the derivative is equal to zero precisely at the places of the graph where the tangent line to the graph has slope zero.

So, just looking at the graph, where does the graph become "flat"? It seems to become flat near at the horizontal coordinates of -2.8 (ish), 0.4 (ish), and 3.3 (ish). Because the problem is asking for an estimate based on what you can see (and not on precise computations), any answer near this range would be acceptable.

Here is a picture showing the points with these three horizontal coordinates – notice they are indeed where the graph looks "flat.'



(b) To find where the derivative is negative, we have to find out where the tangent lines to this graph are have negative slope. The graph begins at the horizontal coordinate -5, and continues to slope downward until around -2.8 (where we estimated the horizontal tangent to be). Then, the graph is sloped downward between the horizontal coordinates 0.4 and 3.3. To give you a visual, let me bold the part of the graph that has negative derivative:



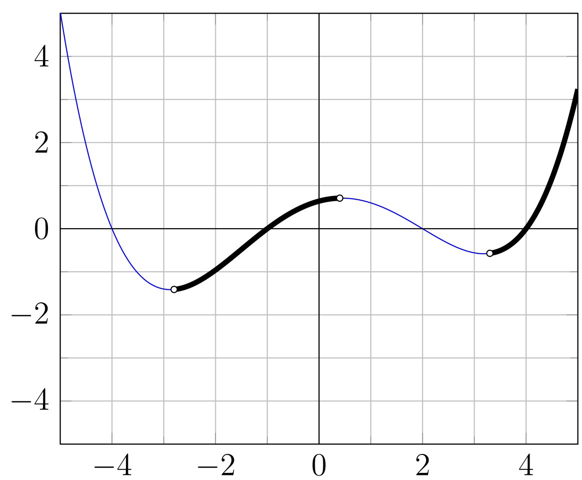
Now, you might notation that there are symbols one can use to describe this portion of a graph – or, better, the portion of the x-axis above which these shaded regions live. Here is one way to answer the question, then, without drawing a picture and without shading:

$$[-5,-2.8)\bigcup(0.4,3.3).$$

Equivalently, you could write

$$\{-5 \le x < -2.8 \text{ or } 0.4 < x < 3.3\}.$$

(c) To find where the derivative is negative, we have to find out where the tangent lines to this graph are have positive slope. I have bolded the part of the graph where this happens:



Here is the notation describing the regions on the x-axis above which the shaded parts live:

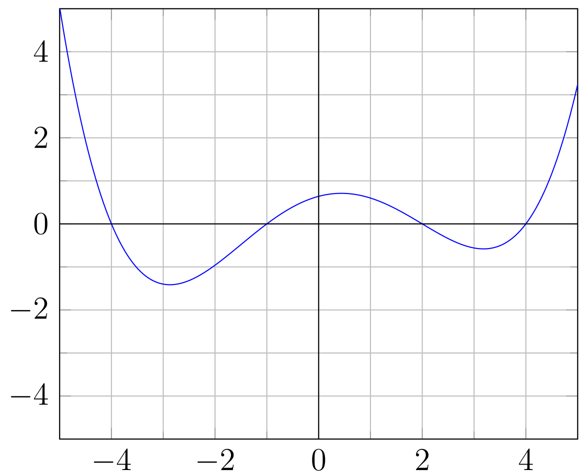
$$(-2.8,0.4)\bigcup(3.3,5].$$

Equivalently, you could write

$$\{-2.8 \le x < 0.4 \text{ or } 3.3 < x \le 5\}.$$

$\square$

Exercise 4.1.3. Below is the graph of the same function $f(x)$ from before.



Sketch a graph of the derivative of f.

Some solutions. Let's remember how to plot *any* function g — for some nice set of horizontal coordinates x, we compute the number $g(x)$. Then we plot the points with horizontal coordinate x and vertical coordinate $g(x)$. When we have some reason to believe that we've plotted enough points to gain salient information, we draw a curve by hand interpolating between the points we've drawn.

In our setting, we're supposed to plot the derivative of f. Thankfully, the previous problems gave us a lot of information about f'.

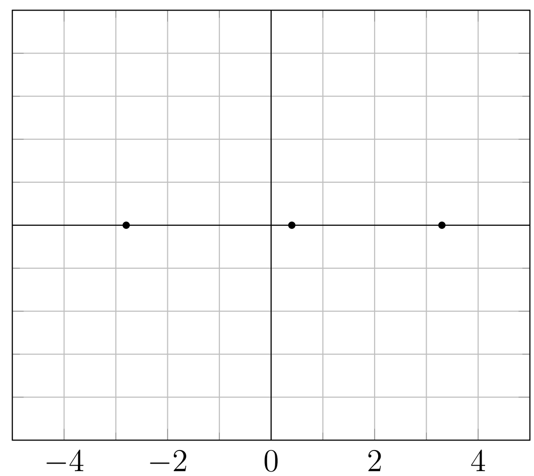
First, we have some idea of where f' equals zero. We decided that f' equals zero roughly around the points where $x = -2.8$, $x = 0.4$, and $x = 3.3$. So let's plot the points

$$(-2.8, f'(-2.8)), \qquad (0.4, f'(0.4)), \qquad (3.3, f'(3.3))$$

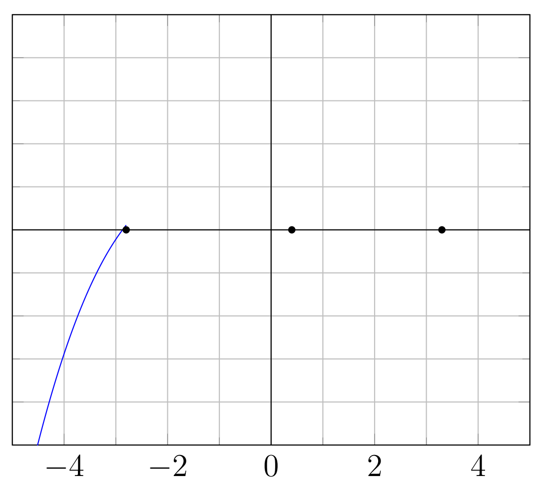
which – because we chose these x-coordinates as places where f' is zero, can be simplified to

$$(-2.8, 0), \qquad (0.4, 0), \qquad (3.3, 0).$$

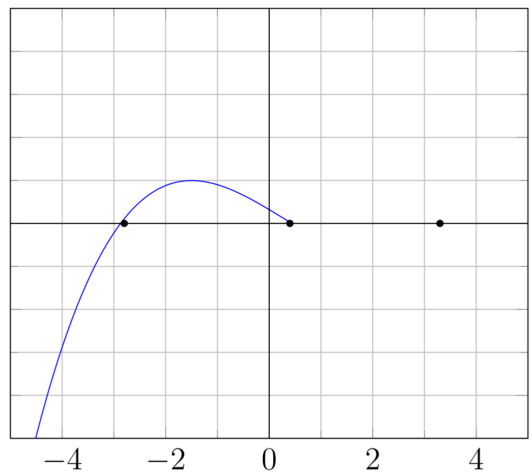
These three points are plotted below:



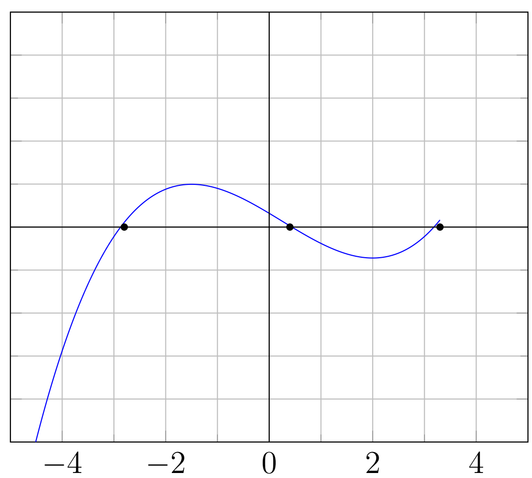
Now, we also saw that from -5 to -2.8, the value of f' is negative. Note also that, in the original graph of f, the slope of the tangent line f is *steeper and steeper* as we move leftward from $x = -2.8$ to $x = -5$. So we can, for the purposes of this sketch, draw a graph that does exactly that – is negative when $-5 \leq x < -2.8$, and has a bigger negative value as x moves toward -5:



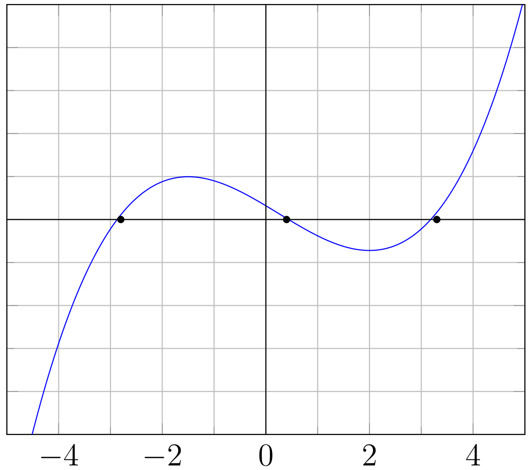
Next, we saw that from -2.8 to 0.4, the value of f' is *positive*. Now, this means we want to draw some curve that goes from $x = -2.8$ to $x = 0.4$ that stays above the x-axis, but goes between the two black dots there. This involves a little bit of guesswork as to what the maximal height that f' attains is. That's okay; let's just draw something that does the job:



Likewise for the interval between $x = 0.4$ and $x = 3.3$, we draw a curve interpolating between the two black dots, but staying below the x-axis (because we know that f' is negative along this region – see our previous problem). Here we go:



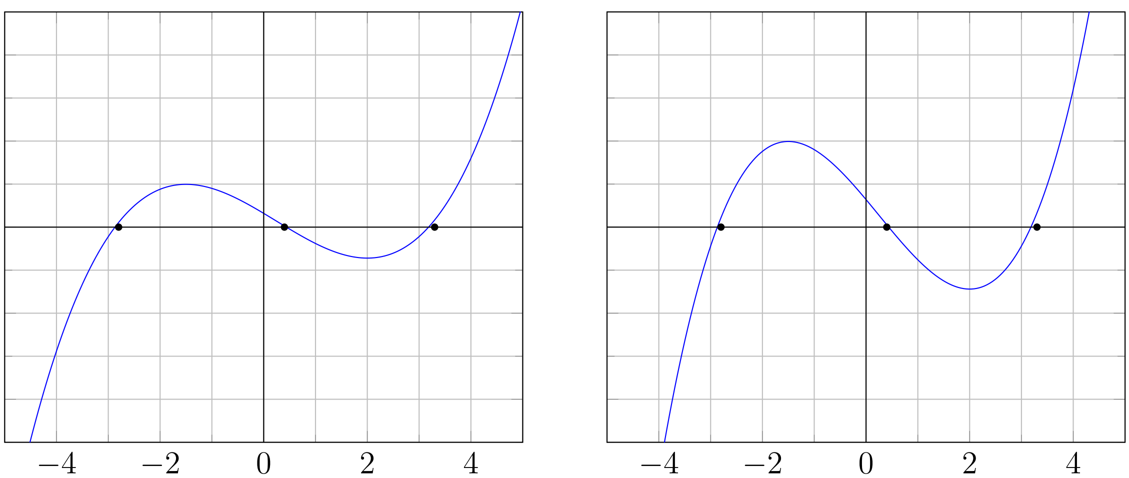
And we finish things off by observing that f has larger and larger positive slope as we move from $x = 3.3$ to $x = 5$, and draw f' accordingly:



(4.1.1)

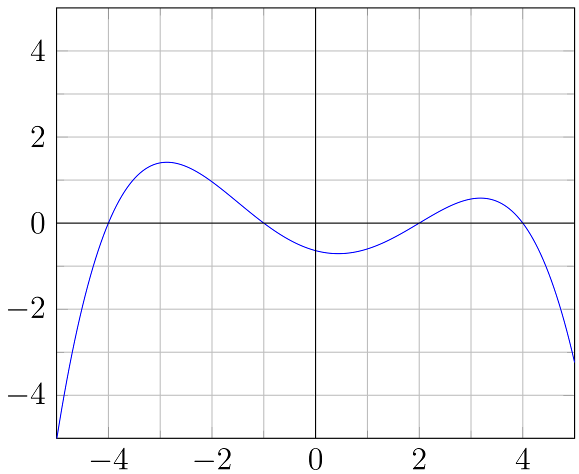
□

Remark 4.1.4. So (4.1.1) is one possible graph you could have drawn. But remember that we had to "guesstimate" how steep a slope f has, and where this slope was steepest. You could have also drawn a sketch of f' that looks like the righthand graph instead:

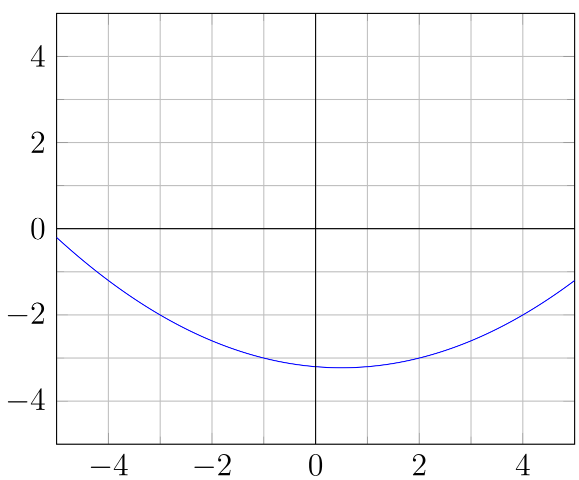


These are genuinely different graphs of f'. For example, the graph on the left has a value of around 1 at $x = -1.5$, while the graph on the right seems to have a value of 2 at $x = -1.5$. So if you and your friend drew these pictures when drawing f', the two of you clearly

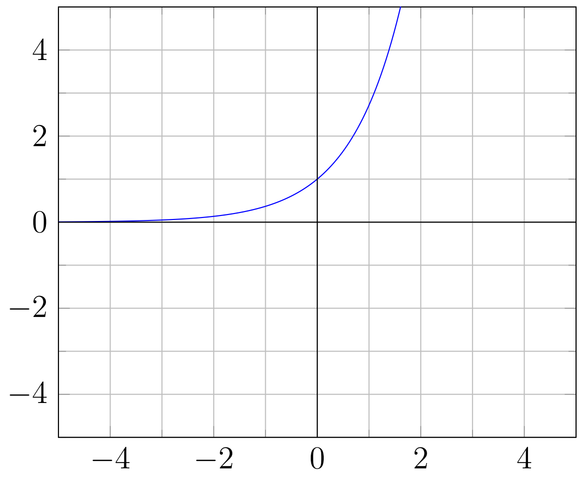
Exercise 4.1.5. Below are graphs of various functions. For each, sketch a graph of its derivative.



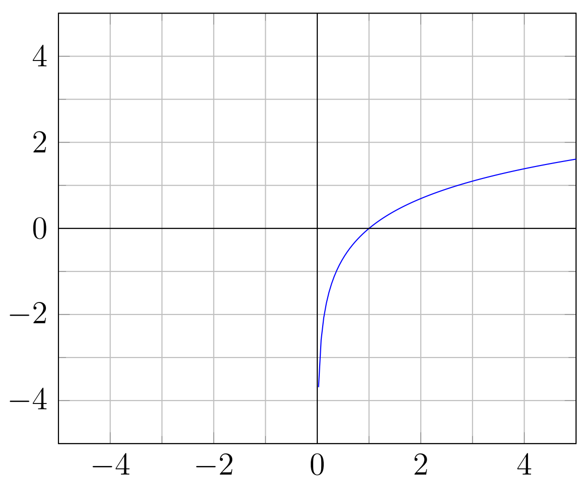
(a)



(b)

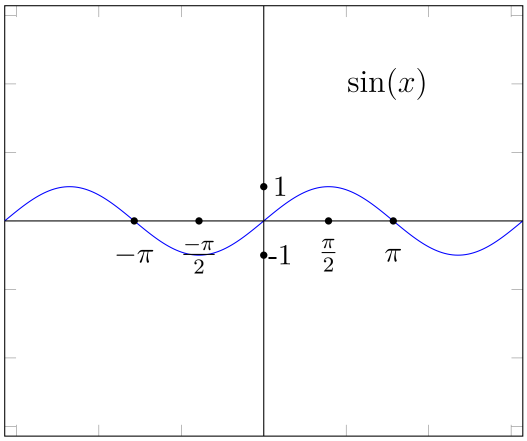


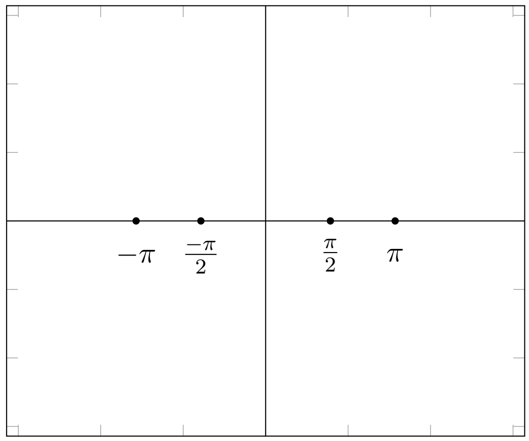
(c)



(d)

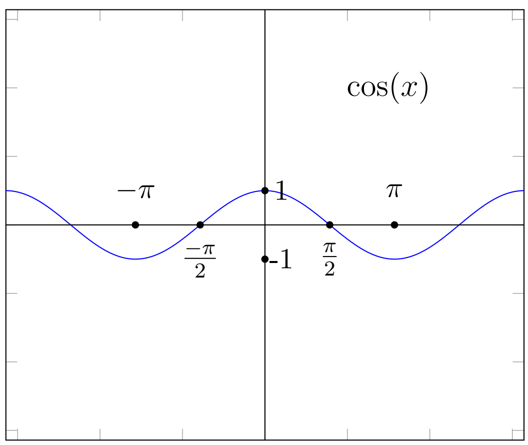
Exercise 4.1.6. Below on the left is the graph of $\sin(x)$.





Based on the lefthand graph, **draw the graph of** $\frac{d}{dx}\sin(x)$ on the right. No need for extreme precision.

Exercise 4.1.7. Below is the graph of $\cos(x)$. Does it have any relation to the picture you drew?



We have seen evidence of the derivative of sine being cosine. This turns out to be true! In fact, let me also tell you what the derivative of cosine is, too:

Theorem 4.1.8 (Derivatives of sine and cosine)**.**

$$\frac{d}{dx}(\sin)(x) = \cos(x), \qquad \frac{d}{dx}(\cos)(x) = -\sin(x).$$

Written another way,

$$(\sin)'(x) = \cos(x), \qquad (\cos)'(x) = \sin(x),$$

or

$$\sin' = \cos, \qquad \cos' = -\sin,$$

or

$$\frac{d}{dx}\sin = \cos, \qquad \frac{d}{dx}\cos = -\sin.$$

In words, *the derivative of sine is cosine. The derivative of cosine is* **negative** *sine.*

Example 4.1.9. Let's find the slope of line tangent to the graph of $\sin(x)$ at $x = \pi$. We know from above that

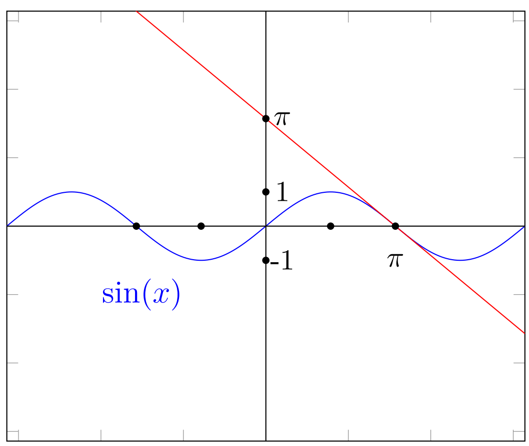
$$(\frac{d}{dx}\sin)(x) = \cos(x),$$

so

$$(\frac{d}{dx}\sin)(\pi) = \cos(\pi).$$

And now we must remember from trigonometry that $\cos(\pi) = -1$.

Here is a picture to confirm that, indeed, the tangent line at $x = \pi$ looks like it has slope -1:



(The tangent line is drawn in red.)

Exercise 4.1.10. Find the derivative of $\cos(x)$ at the following points:

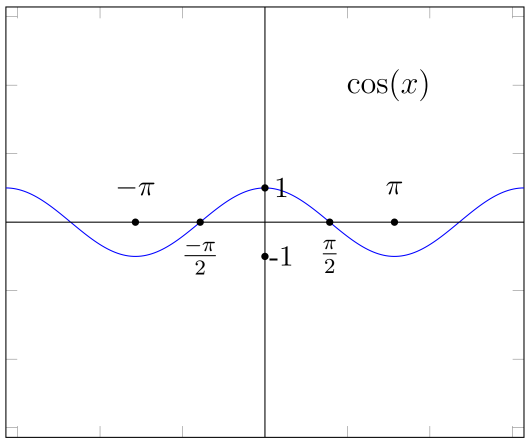
(a) $x = 0$

(b) $x = \pi/2$

(c) $x = \pi$

(d) $x = -\pi$

Do your answers make sense when you look at the graph of $\cos(x)$?



4.2 Bonus: The start of proving that $\frac{d}{dx}(\sin x) = \cos x$.

I want to *prove* to you that the derivative of sine is cosine.

Before that, I need to prove two *lemmas*. A lemma is a statement that's a bit tricky to prove, but that you need to prove in order to prove a theorem.

Lemma 4.2.1.

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

In other words, as h approaches 0, the above fraction $\sin(h)/h$ approaches 1.

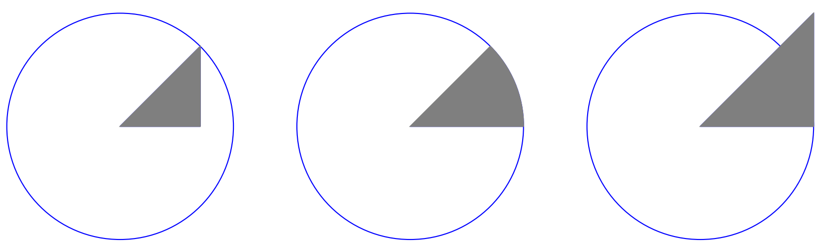
Again, note that we can't just "plug in $h = 0$" to verify this statement – we run into the issue of dividing by zero.

We will prove this by something called the "squeeze theorem." You'll see.

Proof of Lemma 4.2.1. For today, you were prepared with the knowledge that

$$\frac{\sin x \cos x}{2} \leq \frac{x}{2} \leq \frac{\sin x}{2\cos x}. \tag{4.2.1}$$

This inequality is true because these expressions are the areas of the shaded regions below, formed by the angle x:



Note two things: The angle x is *positive* in the pictures above, and also *less than* $\pi/2$—that is, less than 90 degrees. Thus, $\sin x$ is *positive.* So we can divide the inequality in (4.2.5) by $\sin x$ without changing the directions of the inequalities:

$$\frac{\cos x}{2} \leq \frac{x}{2\sin x} \leq \frac{1}{2\cos x}. \tag{4.2.2}$$

Now, note that if I have two numbers a, b satisfying $a \leq b$, then I know that $(1/a) \geq (1/b)$. So, I can "flip the fractions" and flip the inequalities. That is, (4.2.2) implies the following:

$$\frac{2}{\cos x} \geq \frac{2\sin x}{x} \geq 2\cos x. \tag{4.2.3}$$

Dividing everything by 2, I obtain

$$\frac{1}{\cos x} \geq \frac{\sin x}{x} \geq \cos x. \tag{4.2.4}$$

And now I use something called the *squeeze theorem*. This theorem says that, if both the extreme terms in (4.2.4) approach the same value as x goes to zero, then the middle term approaches that value, too.

The outer terms of (4.2.4) have limits as $x \to 0$, and in particular, as x approaches zero from the right[1]:

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\lim_{x\to 0} \cos x} = \frac{1}{\cos(0)} = \frac{1}{1} = 1$$

and[2]

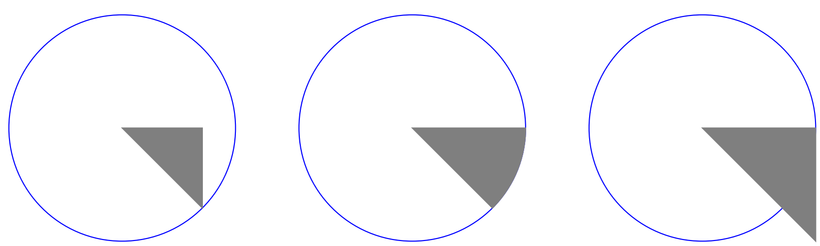
$$\lim_{x\to 0} \cos x = \cos(0) = 1.$$

These two limits agree (they are both 1)! So, by the squeeze theorem, we conclude

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Now, we have to compute the *lefthand* limit[3] of $\sin(x)/x$ and see that it equals 1; then we will have proven the lemma.

Well, when x is negative, the areas of the shaded regions change.



The areas become

$$\frac{-\sin x \cos x}{2} \leq \frac{-x}{2} \leq \frac{-\sin x}{2\cos x}. \tag{4.2.5}$$

(One way to see this is that areas must always be *positive*, so if x is negative, we must flip the sign of $\sin x$ and x to obtain positive numbers in the inequality.)

[1] That is, when we take x to be a positive number shrinking to zero.

[2] Note that we have used our knowledge that cos is continuous to see that $\lim_{x\to 0}\cos(x) = \cos(\lim_{x\to 0} x) = \cos(0)$.

[3] That is, when we take x to be a negative number shrinking to zero.

Now, dividing by $-\sin x$ (which is a *positive* number when the angle is negative!) we obtain (4.2.2) again, and the rest of the algebraic work we did before carries through to show that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1.$$

Thus, we see that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1. = \lim_{x\to 0} \frac{\sin x}{x}.$$

Because the righthand and lefthand limits agree, the limit as $x \to 0$ exists, and we conclude

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

□

You are expected to know this particular limit from now on, and you may use it at will. That is, you are expected to know

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We haven't yet proven that the derivative of $\sin(x)$ is $\cos(x)$. To complete this proof will be your writing assignment for this week!

4.3 For next time

For next time, you should be able to look at a graph of $f(x)$ and:

(a) Draw/sketch a tangent line at whatever point of the graph I ask you about.

(b) If someone gives you a line, be able to tell whether that line looks like it could be a tangent line to the graph.

(c) Estimate where the derivative of f is positive.

(d) Estimate where the derivative of f is negative.

(e) Estimate where the derivative of f is zero.

(f) Tell me the derivative of sin and of cos.