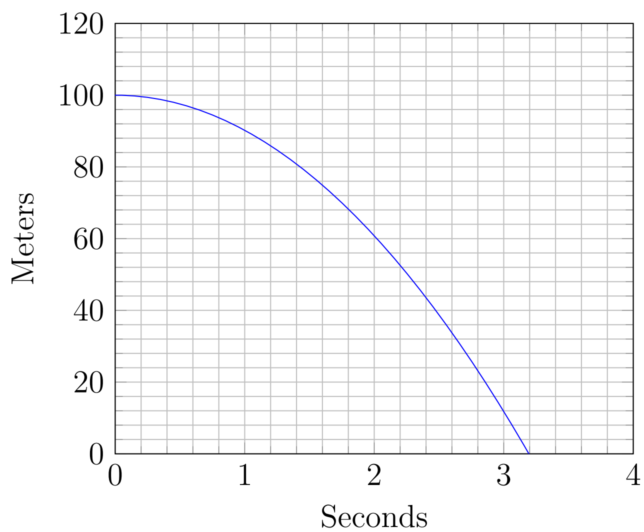
Lecture 2

Secant lines and tangent lines

2.1 Speed of a falling ball

Below is a graph depicting the height $f(t)$ of a ball at time t.[1]



The horizontal axis is in units of seconds, and the vertical axis is measured in meters. You can interpret the graph as depicting what happens over time when you drop a ball from 100 meters high.

[1] This is an *actual* graph of how a ball would fall on the surface of Earth, if dropped in a tube with no air (i.e., in the absence of air resistance).

Exercise 2.1.1. (a) Approximately, what is the height of the ball at time $t = 2$ seconds?

(b) Suppose your friend tells you the height of the ball after $2+h$ seconds.[2] That is, your friend tells you $f(2+h)$. In terms of $f(2)$ and $f(2+h)$, how far does the ball vertically travel between times 2 and $2+h$?

(c) Over that period of time, what is the average speed of the falling ball? (Remember that average speed of an object is given by dividing the distance traveled by the time taken.)

(d) What does this average speed have to do with the line passing through the point $(2, f(2))$ and the point $(2+h, f(2+h))$?

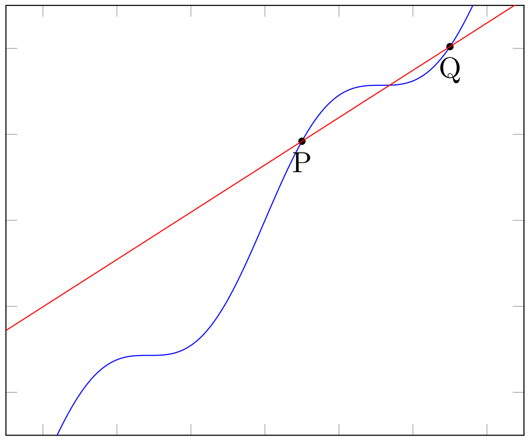
(e) Explore with your group: How could you change h if you want to know the speed of the ball *at* time $t = 2$?

2.2 Secant lines

Let f be a function, and consider the graph of f. Choose two points P and Q on this graph. Then the *secant line through P and Q* is the line passing through P and Q. (Note that a secant line may pass through the graph at more points than just P

[2] Here, h is some number – we don't specify what number it is. Being able to use a symbol like h, instead of demanding a specific number, is one power of *algebra*. By showing some things are true for h regardless of what number it is, we are showing that something is true for *any* value of h. This is immensely powerful.

and Q.)



(2.2.1)

Above, in blue, is the graph of some function f. I chose two points P and Q on this graph, and in red, I drew the line between them. This red line is the secant line passing through P and Q. Note that the secant line *can* pass through more points of the graph of f than just P and Q.

2.2.1 Slopes of secant lines (Intuition)

A secant line allows us to "approximate" a curvy graph using a line instead. For example, (2.2.1) has a complicated blue curve, but after choosing two points on it called P and Q, you can draw a secant line through P and Q. Studying the secant line can give you insight into studying the curve itself.

So, *even when a graph is not a line* (and is curvy), secant lines can help us study the graph using techniques that only apply to lines.

To be honest, secant lines will play an auxiliary role for us, as we will see. They won't come up in explicit computations after this lecture, but they are at the heart of the definition of tangent lines (see Section 2.4).

2.2.2 Slopes of secant lines (Interpretation)

In Exercise 2.1.1, you drew the secant line through two points on the graph from that exercise – a point with x-coordinate 2, and another point with x-coordinate $2 + h$.

You saw that the slope of that line gave you a measure of the average speed at which the ball was falling between time 2 seconds and time $2 + h$ seconds.

So, slopes of secant lines tell us the *average rate of change* between points P and Q.

2.2.3 Slopes of secant lines (Computation)

Suppose you have some graph of some function f, and you choose two points P and Q on it.

Out of tradition, we will call the x-coordinate of P "x" and the x-coordinate of Q "$x+h$." Then to compute the slope of the secant line from P to Q, we compute:

$$\frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}. \tag{2.2.2}$$

Make sure you understand why this is the slope of the secant line – this is the part where you may want to stop reading and try to understand why (2.2.2) is true. It will make an appearance again shortly.

2.3 Group exercises

I want you to get practice with expressions that look like the following:

$$\frac{f(x+h) - f(x)}{h}$$

(This is the slope of the secant line we were dealing with earlier – see (2.2.2).)

Such expressions are called *difference quotients*. There is a reason these are called difference quotients. First, the numerator measures the *difference* in the value of the function at $x+h$, and the value of the function at x. Then, one divides (i.e., takes the *quotient*) of that difference by the distance between x and $x+h$—this is the h in the denominator.

To evaluate difference quotients concretely, you need to know what the function f is. Once you know what f is, you should be able to tell me what the difference quotient is as an expression of x and of h.

In groups, find the difference quotients of the following functions:

1. $f(x) = 5x + 3$
2. $f(x) = x^2 + x$
3. $f(x) = x^3 + 2$
4. $f(x) = x$

Here are some worked-out examples in case they help.

Example 2.3.1. If $f(x) = 2x + 1$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} = \frac{(2(x+h)+1) - (2x+1)}{h} &= \frac{2x + 2h + 1 - 2x - 1}{h} \\ &= \frac{2h}{h}.\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, the difference quotient equals 2.

Example 2.3.2. If $f(x) = x^2 + 10$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} = \frac{((x+h)^2 + 10) - (x^2 + 10)}{h} &= \frac{x^2 + 2xh + h^2 + 10 - x^2 - 10}{h} \\ &= \frac{2xh + h^2}{h}\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, it equals $2x + h$.

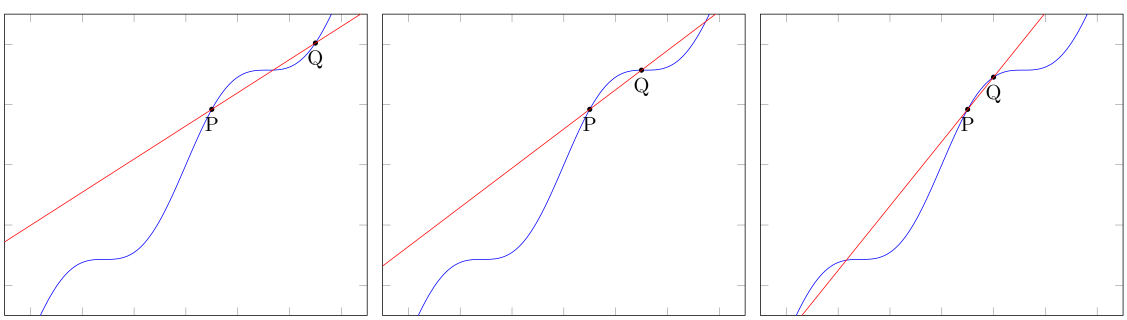
Example 2.3.3. If $f(x) = x^3$, then

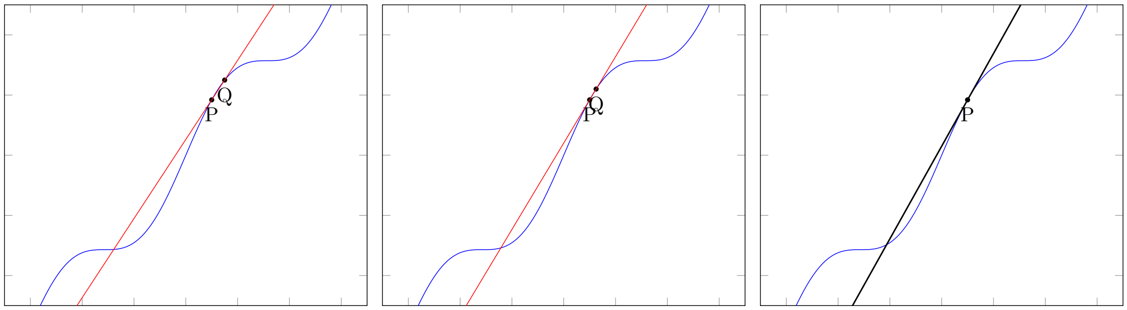
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \frac{3x^2h + 3xh^2 + h^3}{h}\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, it equals $3x^2 + 3xh + h^2$.

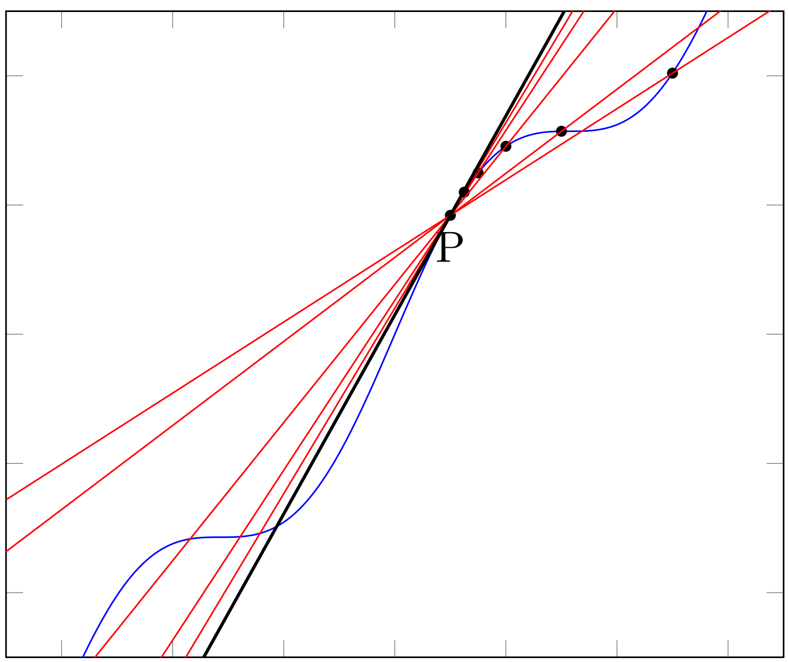
2.4 Tangent lines

Sometimes, if you move the point Q to be closer and closer to P (from either side of P—i.e., from the left or from the right) the secant lines through P and Q approach a line that barely "kisses" the graph of f at P. If such a line exists, we call the line the *tangent line* to the graph of f at P.





Above is a sequence of pictures. In blue is the graph of a function f. We choose a point P, and P doesn't move. But we move a point called Q closer and closer to P. Then the red secant line through P and Q changes (because Q is moving), and the red lines themselves approach the black line in the last image. In the above sequence of pictures, Q is approaching P from the right; it turns out that if we were to make Q approach P from the left instead, the secant lines would still approach the black line. So, following the definition of "tangent line" I just gave, we see that the black line is the tangent line to the graph of f at P.



In this bigger picture, we combine the "movie" of the secant lines (drawn in red) as they approach the tangent line at P (drawn black).

2.4.1 Slopes of tangent lines (Intuition)

If a graph of a function f has a tangent line at a point P, the tangent line is the line that "best approximates" the graph out of all other possible lines that pass through P. Oftentimes, it is the line that barely "kisses" the graph there.

Another intuitive way to think about the tangent line at P is as follows. Pretend that the graph of a function f is the path of a car. Then the tangent line to f at P traces out the directions in which the headlights and backlights of the car point when the car is at point P.

2.4.2 Slopes of tangent lines (Interpretation)

We saw that slopes of secant lines told us about an "average rate of change" between two points P and Q. When we make Q approach P, the slope of the secant line approaches the slope of the tangent line at P — we interpret the slope of the tangent line as an *instantaneous* rate of change. This slope no longer depends on choosing two points P and Q to measure an average over; it only depends on how the function behaves near P.

So while a secant line to a position-versus-time graph would tell us the average speed of a moving object between two points in time, a tangent line at a given moment will tell us the speed – at that moment. (No averages are needed.)

2.4.3 Slopes of tangent lines (Computation)

Again, let us denote the horizontal coordinate of the point P by x. If we choose Q to be a point with horizontal coordinate given by $x + h$, the slope of the secant line was given by (2.2.2). Because the tangent line is obtained by moving Q closer and closer to P, its slope is obtained by asking what happens to (2.2.2) as h approaches zero. (Make sure you understand the previous sentence – it entails understanding what h stands for.)

If there is a number to which (2.2.2) approaches as h goes to zero, we call this number the "limit" of the expression as h goes to zero. We denote this limit as follows. (This is a new mathematical expression, so you may want to spend some time understanding what it means.)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.4.1}$$

In other words, understanding the slope of a tangent line at P is a two-step process. You first have to understand the faction (2.2.2). Then, you have to understand what happens to this fraction as h goes to 0.

Warning 2.4.1. The expression (2.4.1) is *not* the same as what happens when you "set $h = 0$." This is one of the subtleties of the game. After all, because you are dividing by h in the fraction, you cannot set $h = 0$. (You cannot divide by zero.)

This distinction is at the heart of the progress that made calculus possible: We often cannot set a variable to equal a particular value, but we can still ask how a function behaves as that variable *approaches* the value we're interested in.

2.5 For next time

For next class, you should be able to simplify the difference quotient (as a function of h) for the following functions f and values of x:

1. $f(x) = 3x + 2$, with $x = 3$.
2. $f(x) = x^2$, with $x = 2$.
3. $f(x) = x^2 + 1$, with $x = -1$.

You should also be able to tell me what x and h have to do with the points P and Q we've talked about. You should also be able to explain what the difference quotient has to do with the secant line through P and Q. Finally, you should be able to say what a tangent line to a graph at a point with horizontal coordinate x is.