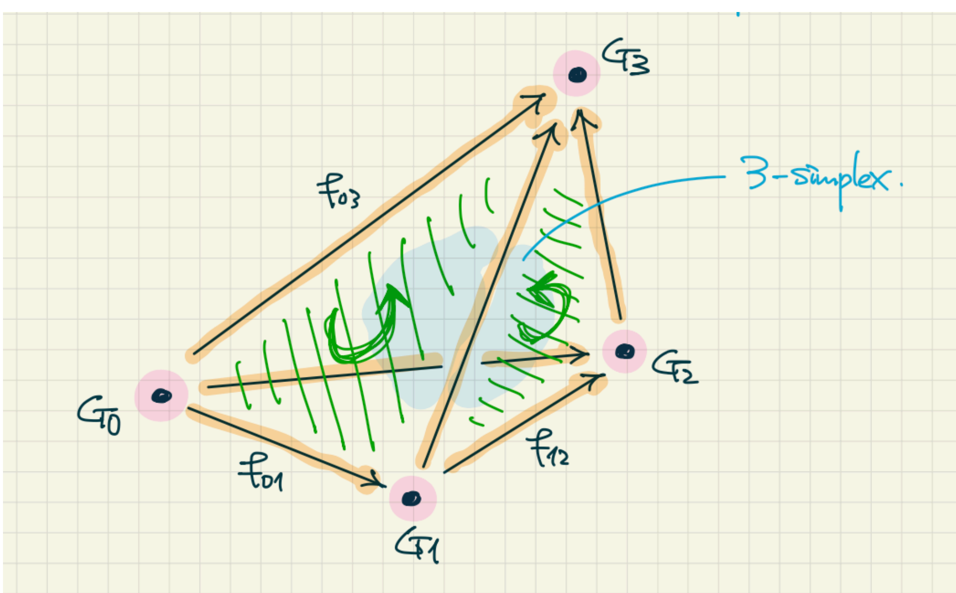
Higher algebra in geometry



Notes from a Summer 2023 lecture series

Notes by Masato Tanabe

Lectures by Hiro Lee Tanaka

Draft: August 2023

Preface

These are hand-written notes taken by Masato Tanabe based on lectures delivered by myself in August of 2023 in Wakou City, Japan at RIKEN iTHEMS. Some spoken commentary did not make it into the hand-written notes, so for these, I refer the reader to the introductory text of each lecture. The present work also contains some exercises.

I wanted to do something different from the previous times I was asked to give a lecture series [14, 15, 16], so in these lectures I devoted the first week to delving more into the combinatorial underpinnings of infinity-categories (also known as quasi-categories after Joyal [5], and as weak Kan complexes after Boardman-Vogt [2]). I tried to teach the subject the same way I learned it – without knowing any category theory, and through combinatorial intuitions. The aim was to lower the bar for entry.

In the second week I presented some applications to the study of smooth manifolds and to symplectic/contact geometry. I tried when possible to illustrate why and how the theory of infinity-categories was useful in solving or capturing concrete geometric problems. I presumptuously talked only about results in which I was somehow involved. My intent was to be honest when infinity-categories are and are not so necessary.

We hope the reader will be left wanting more. Lurie's Higher Topos Theory [8] and Higher Algebra [9] are the canonical references. I would in fact recommend beginning with Higher Algebra. By studying the proofs there (which faithfully cite the results of Higher Topos Theory) the technical ingredients of [8] gain context. An evolving and highly readable reference, also due to Lurie, is Kerodon [10].

These lectures would not have been possible without the significant hospitality and efforts of the organizer, Taketo Sano, and of the staff of iTHEMS RIKEN, especially Chika Oota. I would like to thank them for their generosity. I would also like to thank both the online and in-person participants for their questions and engagement.

I give my thanks of course to Masato Tanabe for his wonderful notes.

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Contents

Lecture I

Introduction to higher algebra for a general audience

I was asked to open the lectures by introducing our main topic to scientists without a background in pure mathematics.

I emphasized that associativity is an amazing property of multiplication. (Imagine having to parenthesize 100 matrices carefully before a computer can compute their product.) At the same time, I disambiguated commutativity from associativity, giving an example – Rock Paper Scissors, and more generally, competitions with 2-to-1 operations indicating winners – where associativity fails but commutativity holds.

I then gave concrete examples of the notion of homotopy, and of algebraic structures that only hold up to homotopy. The fundamental group was the first example.

From theoretical physics (not "twisted" physics), the A model gives rise to the Fukaya category. Here it was emphasized that being associative up to homotopy is only so useful – for when deciding how to multiply 4 elements a, b, c, d one finds two non-canonical homotopies relating $(a(bc))d$ to, say, $(ab)(cd)$. This is drawn as the boundary of a pentagon in the notes, where the edges of the pentagon are known homotopies coming from three-term associativity. Thus one should – if one wants to unambiguously know why computations give equivalent answers up to known homotopies – demand extra data that exhibits these two non-canonical homotopies as themselves homotopic. This higher homotopy is the interior of the pentagon. This of course does not stop with four-term associativity; one will find a three-cell to be necessary to unambiguously associate five terms, and so on. The

Fukaya category supplies such cells for arbitrarily many terms, and thus is an example of a so-called A_∞-category (as opposed to, say, A_4) for this reason.

The final example arose from a factorization-algebra-esque picture of a topological quantum field theory. If one imagines a field theory to be able to assign observables to every patch of space-time (drawn as $\mathbb{R}^2$ in the notes), and if the observables only depend on the topological type of the patch of space-time, one finds that the collection of observables has a commutative structure that is only commutative up to non-canonical homotopies. In a two-dimensional theory, this non-canonicity is detected by the presence of winding numbers. In three dimensions, this winding number obstruction vanishes, but there are inequivalent hemispheres nullifying winding numbers; thus a π_2 obstruction to commutativity appears. Such a structure in n dimensions is called an E_n-algebra.

As a preview, I stated that the simplest version of factorization homology allows us to create n-manifold invariants out of E_n-algebras. In fact, if one takes E_n-algebras in a symmetric monoidal ∞-category with sifted colimits, and wherein the symmetric monoidal structure commutes with sifted colimits in each variable, every multiplicatively local-to-global invariant (i.e., every $\otimes$-excisive invariant) arises uniquely from an E_n-algebra.

2023-7-31 (月)

Higher Algebra in Geometry ①

{ Today HIGHER ALGEBRA }

① Associativity

$$(2\times 3)\times 5 \overset{\text{Miracle}}{\equiv} 2\times(3\times 5)$$

Something NOT associative : ジャンケン！

$$RR = R, \quad PP = P, \quad SS = S,$$
$$RP = PR = P,$$
$$RS = SR = R,$$
$$SP = PS = S.$$

(This multiplication is commutative)

→ 勝ち抜き戦 :

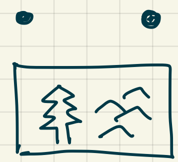
$$R(PS) = RS = R$$
$$(RP)S = PS = S$$

different!

∴

{ 推移性のない"順序"から作っていると見做せる. }

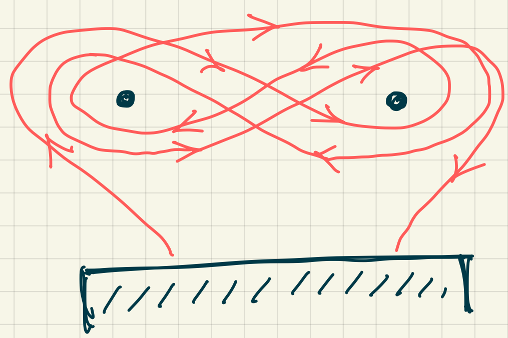
② Homotopy



Puzzle :

Can you hang (using string) this picture
on these two nails
such that if any one nail is removed,
this picture falls ?

my answer →



Def. (informal)

A **homotopy** (from one configuration to another)
is a continuous deformation (between the two configurations).

Higher Algebra :

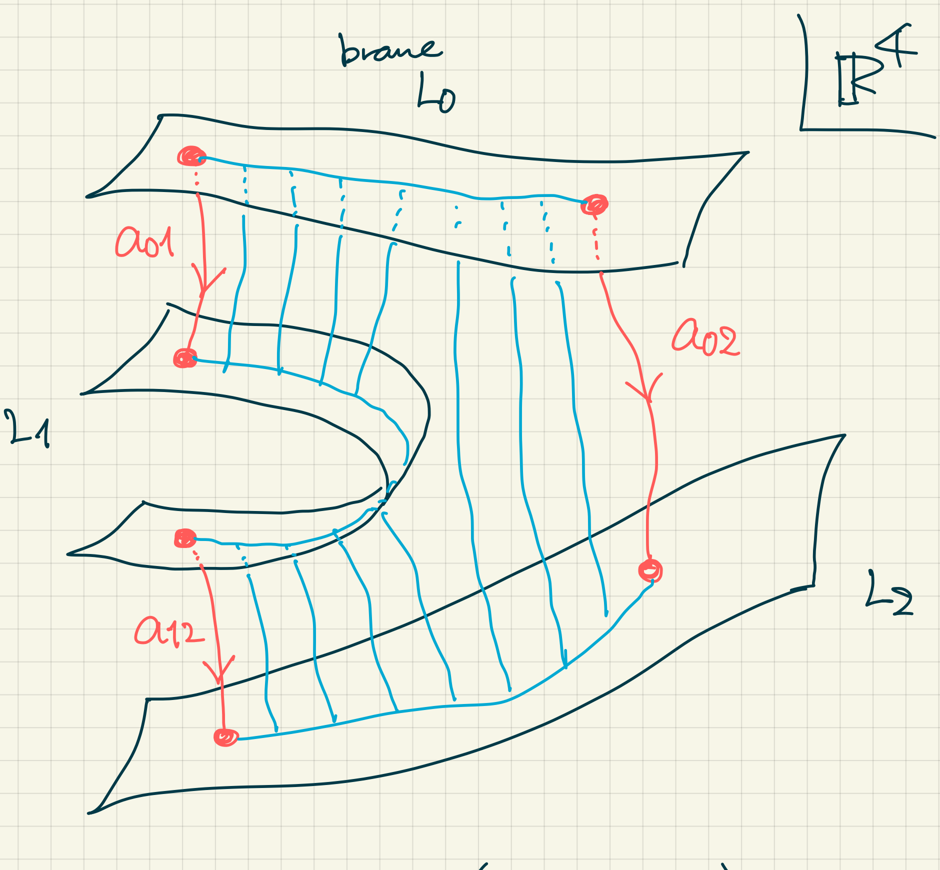
Where homotopy coexists with associativity.

Theorem

The operation of concatenating loops is associative up to homotopy.

☆ Another example (Twisted physics)

Topological twist of a supersymmetric string theory.



$$a_{12} \cdot a_{01} := \sum_{a_{02}} \left(\# \text{ of } \right) \cdot a_{02} .$$

<u>Theorem</u> This is associative up to homotopy.

($\to$ The Fukaya category)

$$(ab)c \sim a(bc).$$

$$\Downarrow$$

$$(a(bc))d \sim a((bc)d)$$

$$((ab)c)d \qquad a(b(cd))$$

$$(ab)(cd)$$

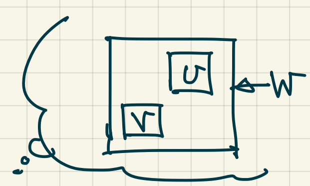
☆ A third example

Given a physical theory
that can measure/observe things in 2-D,
let

これを観測する.

$$\mathrm{Obs}(U) := \{ U\text{内の観測器} \}.$$

in $\mathbb{R}^2$



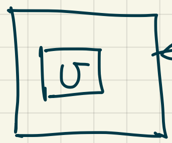
$\rightsquigarrow$ We have an operation

$$\mathrm{Obs}(U) \times \mathrm{Obs}(V) \longrightarrow \mathrm{Obs}(W).$$

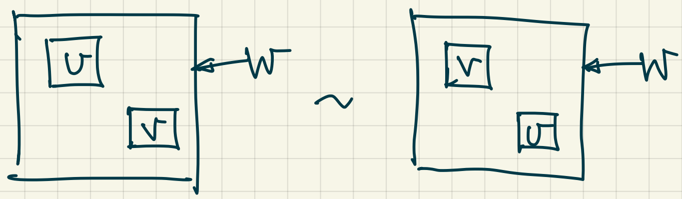
Let's suppose there are collection,

$$\mathcal{F}(U) \subset \mathrm{Obs}(U)$$

of topological observables.

⇝ 

$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is an equivalence.

Now, play 

$$A \times A \longrightarrow A$$

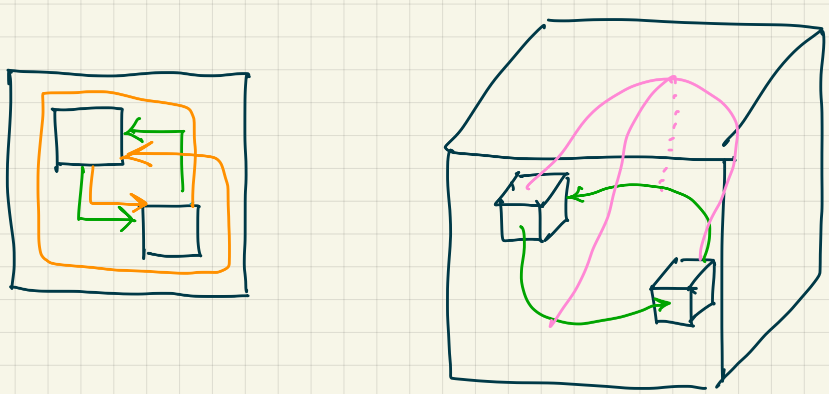
$$\rightsquigarrow \begin{tikzcd} \mathcal{F}(U) \times \mathcal{F}(V) \arrow[r] \arrow[dr, phantom, "\wr"] & \mathcal{F}(W) \\ \mathcal{F}(V) \times \mathcal{F}(U) \arrow[r] & \mathcal{F}(W) \end{tikzcd}$$

$$A \times A \longrightarrow A$$

(commutative !)

$(\mathcal{F}(U) \cong \mathcal{F}(V) \cong \mathcal{F}(W) =: A)$

<u>Remark</u> This commutativity is true for all dimensions $\geqslant 2$.



<u>Remark</u> These "n-dimensional topological algebras" is called $\mathbb{E}_n$-algebras.

Theorem (Ayala–Francis, Ayala–Francis–T.)

(informal:)

The collection of $\mathbb{E}_n$-algebras
is equivalent to
the collection of invariants of n-mfd.s $\square$
that are local–to–global and multiplicative.

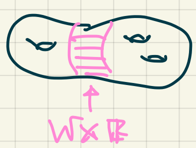
some tangencial conditions.

FACTORIZATION HOMOLOGY

Given two mfd.s, we can make

$$X \amalg Y \xmapsto{\text{multip.}} Z(X) \times Z(Y).$$

$$X \# Y \longmapsto Z(X) \times_{Z(W \times \mathbb{R})} Z(Y)$$



Theorem

Let $\mathcal{C}$ be a precortable ∞-category w/ symmetric $\otimes$ structure connecting w/ sifted colimits in each variable.

$\Rightarrow$ $\exists$ an equiv. of ∞-cat.s

$$\mathbb{E}_n\text{-alg}(\mathcal{C}^{\otimes}) \simeq \mathrm{Fun}^{\otimes}_{\text{Excisive}}(\mathrm{Mfd}_n^{\boxed{fr}}, \mathcal{C})$$

Lecture II

Infinity-categories

Here I tried to convey that you do not need to know much category theory to begin the study of infinity-categories. Indeed, to any natural setting with composition of maps, there are natural data (called commutative diagrams) that admit a combinatorial description. This naturally led us to the notion of simplicial set and of the nerve of a category.

I gave two ways to think about a simplicial set (using generators-and-relations, or as a functor out of a category of linearly ordered sets). From either perspective, it is easy to state what a map of simplicial sets is.

We studied examples: The n-simplex Δ^n, explicitly computing Δ^0 and Δ^1. We also studied products of simplicial sets, with the example of $\Delta^1 \times \Delta^1$ illustrating the power of the degenerate simplices.

After defining horns and the nerve of a category, I asserted that the theory of classical categories is equivalent to the theory of simplicial sets satisfying some unique-horn-filling condition. This is one sense in which combinatorics may completely replace a theory of categories.

Then we saw that topological spaces give rise to simplicial sets by the singular complex construction. Such simplicial sets satisfied a different horn-filling condition, rendering them Kan complexs. In fact, mentioned in the lectures was a fundamental fact of life: To a homotopy theorist, a topological space may as well be a Kan complex (and vice versa).

Studying properties common to both the above examples, we defined the notion of an infinity-category. I emphasized that the definition is purely combinatorial. We saw the notion of functor, and of natural transformation of functors.

We recommend Chapter 1 of Higher Topos Theory [8] for further reading.

2023-8-1 (火) HA in G ②

(Last time : Intro to HA.

Today : ∞-categories)

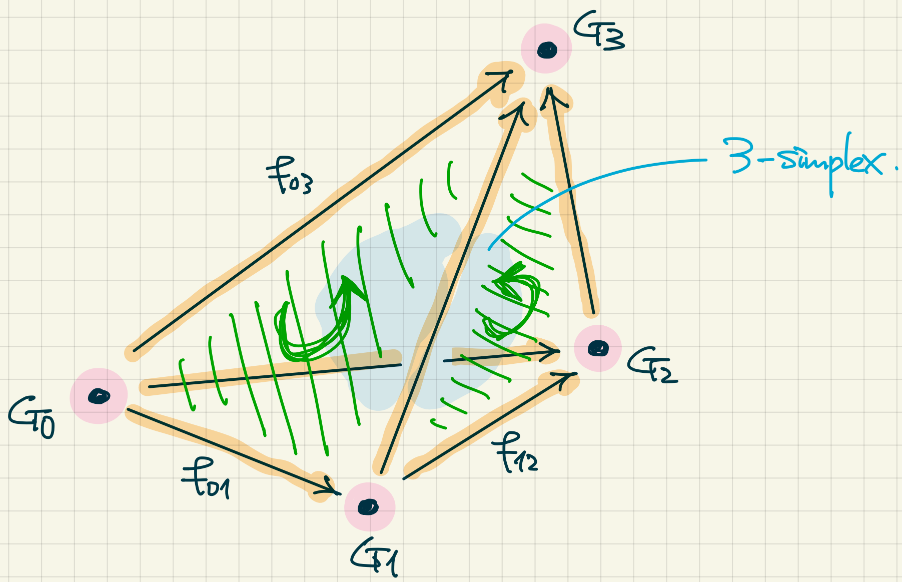
Example | exploration : The category of groups.

To "draw" this category:

For every group G, draw a vertex (or mashmallow);

For every group hom. $f : G \to H$,
draw an oriented edge.

Every time we have a commutative triangle [n-simplex]
$g \circ f = h$, draw an oriented triangle. [n-simplex]



<u>Rmk</u> : Any n-simplex is uniquely determined by $f_{01}, f_{12}, f_{23}, \dots, f_{(n-1)n}$.

We can do this for any category, and obtain the following data :

$$\begin{tikzcd} X_0 := \text{the set of vertices} \\ X_1 := \text{the set of (ori.) edges,} \arrow[u, shift left, "d_0"] \arrow[u, shift right, "d_1"'] \end{tikzcd}$$

$$X_0 \xrightarrow{s_0} X_1$$

where

$$\begin{tikzcd} f : G_0 \to G_1 \arrow[d, mapsto, "d_0"'] \arrow[d, mapsto, shift left=8, "d_1"] & \mathrm{id} : G_0 \to G_0 \\ G_1 \qquad G_0 & G_0 \arrow[u, mapsto, "s_0"'] \end{tikzcd}$$

the index of forgotten group

Moreover ...

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] & \cdots\cdots \end{tikzcd}$$

where

$$\begin{tikzcd} & G_2 & & & G_2 \\ G_0 \arrow[r, "f_{01}"'] \arrow[ur, "f_{02}"] & G_1 \arrow[u, "f_{12}"'] & \overset{d_1}{\longmapsto} & G_0 \arrow[ur, "f_{02}"] \end{tikzcd}$$

etc.

@ These functions d_i, s_i satisfy some relations called the simplicial relations :

$$
\left.
\begin{aligned}
&\bullet\ d_i s_j = \cdots = s_k d_l \\
&\bullet\ d_i d_j = d_k d_l \\
&\circ\ s_i s_j = s_k s_l
\end{aligned}
\right\} (\star)
$$

Def.

A simplicial set is the data of

sets $X_0, X_1, X_2, \cdots$ and

functions $s_j : X_i \to X_{i+1}$ $(0 \leq j \leq i)$
$\qquad\qquad d_j : X_i \to X_{i-1}$ $(0 \leq j \leq i)$

satisfying $(\star)$.

★ Alternative description ! :

For every $n \geq 0$,

$[n] := \{ 0 \leq 1 \leq \cdots \leq n \}$: linearly ordered.

⇝ We have :

$$
\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \arrow[r, shift left=4, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_2"'] & {[2]} \arrow[l, shift right=2, "\sigma_0"'] \arrow[l, shift left=2, "\sigma_1"] & \cdots\cdots
\end{tikzcd}
$$

$\left(\begin{array}{l}\sigma_0(0)=\sigma_0(1)=0,\\ \sigma_0(2)=1.\end{array}\right)$

Define a category Δ as :

$$\mathrm{Ob}(\Delta) := \{[0],[1],[2],\dots\};$$

$$\mathrm{Hom}_\Delta([m],[n]) := \{ f:[m]\to[n] \mid i\le j \Rightarrow f(i)\le f(j)\}.$$

weakly order preserving.

<u>Def'</u>

A simplicial set is a functor $\begin{array}{ccc}\Delta^{op} & \longrightarrow & \mathrm{Sets}\\ \in & & \in \\ [n] & \longmapsto & X_n.\end{array}$

☆ A map $f: X\to Y$ of simplicial sets is a collexion $\{f_i : X_i \to Y_i\}_{i\ge 0}$

s.t. $\begin{cases} f_{i-1}\circ d_j = d_j \circ f_i \\ f_{i+1}\circ s_j = s_j\circ f_i. \end{cases}$

<u>Eg.</u>

① Given a cat. $\mathcal{C}$, define

$$N(\mathcal{C})_n := \{\text{comm. } n\text{-spx. in } \mathcal{C}\}.$$

<u>Nerve</u> of $\mathcal{C}$.

② Fix $n\ge 0$, and consider

$$\Delta^m([n]) := \mathrm{Hom}_\Delta([n],[m]),$$

$$d_i := \partial_i^*, \quad s_i := \sigma_i^*.$$

$n=0$

$$\begin{array}{cccccc} \Delta^0([0]) & & \Delta^0([1]) & & \Delta^0([2]) & \\ \| & & \| & & \| & \\ X_0 & \underset{s_i}{\overset{d_i}{\leftleftarrows}} & X_1 & \underset{s_i}{\overset{d_i}{\Lleftarrow}} & X_2 & \cdots \\ \| & & \| & & \| & \\ * & & * & & * & \end{array}$$

$n=1$

$$\begin{array}{ccccc} \Delta^1([0]) & & \Delta^1([1]) & & \Delta^1([2]) \\ \| & & \| & & \| \\ \{0,1\} & \leftleftarrows & \{00,01,11\} & \Lleftarrow & \{000,\ \underline{001},\ 011,\ 111\} \end{array}$$

meaning: $\begin{array}{c} 0 \mapsto \\ 1 \mapsto \\ 2 \mapsto \end{array} \boxed{\begin{array}{c} 0 \\ 0 \\ 1 \end{array}}$

③ If X, Y are simplicial sets, define

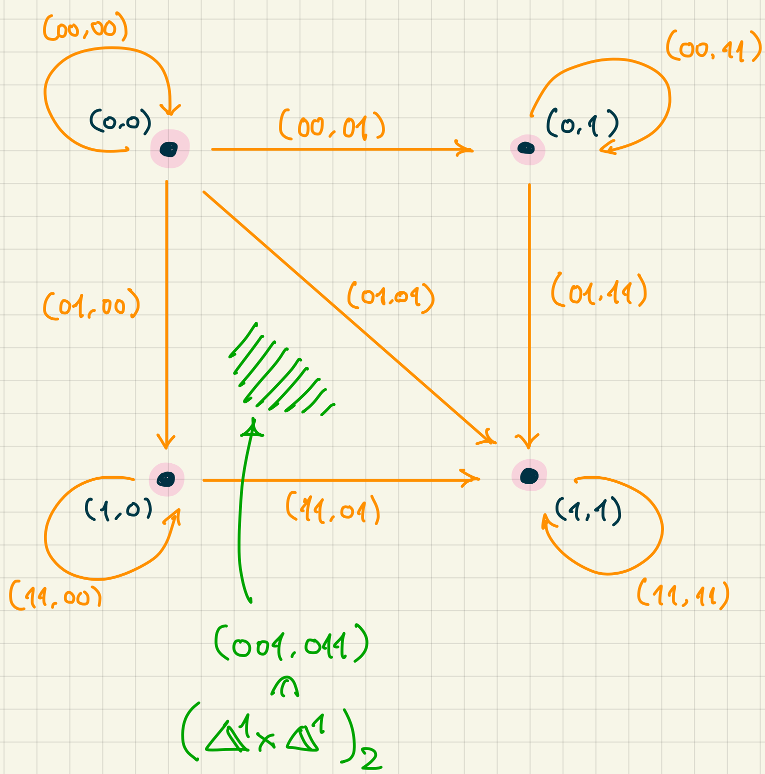
$$(X \times Y)_n = (X \times Y)[n] := X_n \times Y_n,$$

$$d_i := d_i^X \times d_i^Y, \quad s_i := s_i^X \times s_i^Y.$$

e.g.

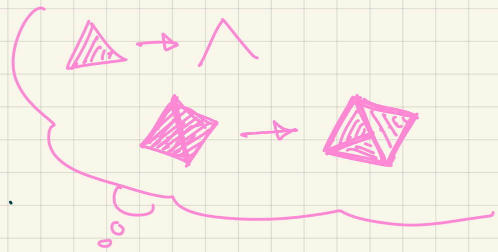
$$(\Delta^1 \times \Delta^1)_0 = \{(0,0),(0,1),(1,0),(1,1)\}.$$

$(\Delta^1 \times \Delta^1)_1 = \{(00,00), \dots\}$ ← 9 elements.



{ Easy combinatorics ! }

(not need any difficult math.)



"Def."

Fix $n \geq 0$ and $0 \leq i \leq n$.

Define Λ^n_i to be the simplicial set obtained from Δ^n by removing $d_i(\Delta^n)$ and $\mathrm{Int}(\Delta^n)$.

i.e. $\Lambda^n_i([k]) := \left\{ [k] \xrightarrow{f} [n] \;\middle|\; \begin{array}{l} f([k]) \text{ does not} \\ \text{contain } [n] \setminus \{i\} \end{array} \right\}$.

E.g. $\Lambda^0_0 = \phi$. $\Lambda^1_0 \cong \Delta^0 \cong \Lambda^1_1$.

Λ^2_0 :
$$\begin{tikzcd} & 1 \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Λ^2_1 :
$$\begin{tikzcd} & 1 \arrow[dr] \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Λ^2_2 :
$$\begin{tikzcd} & 1 \arrow[dr] \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

E.g. A map $\Lambda^2_1 \to N(\mathcal{C})$ is

$$\begin{tikzcd} & C_1 \arrow[dr, "f_{02}"] \\ C_0 \arrow[ur, "f_{01}"] & & C_2 \end{tikzcd} ;$$

A map $\Lambda^2_0 \to N(\mathcal{C})$ is

$$\begin{tikzcd} & C_1 \\ C_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & C_2 \end{tikzcd} .$$

Thm. A simplicial set X is isom. to a nerve of a category

$$\Updownarrow \; !$$

$$\forall\, 0<i<n, \quad \begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Moreover, $X \cong N(\mathcal{C})$, $Y \cong N(\mathcal{D})$

$$\Rightarrow \{ f : X \to Y \} \cong \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

(E.g. continued) 4

Let W be a top. sp.

$\Delta^k := \{ (t_0, \dots, t_k) \in \mathbb{R}^k \mid \sum t_k = 1 \}$

$$\rightsquigarrow \mathrm{Sing}(W)_k := \{ \Delta^k \xrightarrow[\text{cont.}]{} W \}.$$

$\mathrm{Sing}(W)$ is a simp. set !

Rmk : $0 \leq {}^\forall i \leq {}^\forall n$,

$$\begin{tikzcd} \bigcup_{j \neq i} \Delta^{n-1} \arrow[d, equal] \arrow[dr, "\bigcup_{j\neq i} \partial_j"] & \\ \Lambda^n_i \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Def. Any simp. set X satisfying

$$0 \leq \forall i \leq \forall n, \quad \begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

is called a Kan cpx.

Rmk. (Quillen)

The h.topy theory of Kan cpx.es is equivalent to the h.topy theory of spaces.

Def. A simp. set X is called an ∞-category if

$$0 < \forall i < \forall n, \quad \begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} .$$

@ "∞-cat." = "weak Kan cpx." = "quasi-cat."

Lurie Boardman–Vogt Joyal

☆ A functor of ∞-cat.s is a map of simp. sets.

<u>Sketch</u> : The ∞-category of spaces. "$\mathcal{T}op$"

$$\mathcal{T}op_0 := \{ \text{top. sp.s } X \ (\simeq^{\exists} \text{CW cpx.}) \}$$

$$\mathcal{T}op_1 := \{ f_{01} : X_0 \to X_1 \ : \ \text{cont.} \}$$

$$\mathcal{T}op_2 := \left\{ (H, f_{01}, f_{12}, f_{02}) \ \middle| \ \begin{tikzcd} & X_2 \\ X_0 \arrow[ur, "f_{02}"] \arrow[r, "f_{01}"'] & X_1 \arrow[u, "f_{12}"'] \arrow[ul, phantom, "H"] \end{tikzcd} \right\} .$$

Def. Fix $f, g : \mathcal{C} \to \mathcal{D}$.

A natural transformation from f to g

is $F : \mathcal{C} \times \Delta^1 \to \mathcal{D}$

s.t. $\begin{cases} F|_{\mathcal{C} \times \{0\}} = f \, ; \\ F|_{\mathcal{C} \times \{1\}} = g \, . \end{cases}$

Exercises: Basics of ∞-categories

II.1 Homotopy

Fix two arrows $f, g : X \to Y$ in a simplicial set $\mathcal{C}$. (Formally: Choose two elements of $\mathcal{C}_1$ whose images under d_0 agree, and whose images under d_1 agree.)

Here are four ways to define a homotopy between f and g:

- (L) There exists a 2-simplex L in $\mathcal{C}$ such that
$$d_0 L = s_0 Y, \qquad d_1 L = g, \qquad d_2 L = f.$$
- (L') There exists a 2-simplex L' in $\mathcal{C}$ such that
$$d_0 L = s_0 Y, \qquad d_1 L = f, \qquad d_2 L = g.$$
- (R) There exists a 2-simplex R in $\mathcal{C}$ such that
$$d_0 L = g, \qquad d_1 L = f, \qquad d_2 L = s_0 X.$$
- (R') There exists a 2-simplex R' in $\mathcal{C}$ such that
$$d_0 L = f, \qquad d_1 L = g, \qquad d_2 L = s_0 X.$$

Example II.1.0.1. In case it helps, here is a picture of (for example) a simplex L:

$$\begin{tikzcd} & & Y \\ X \arrow[urr, "g"] \arrow[r, "f"'] & Y \arrow[ur, Rightarrow, no head] & \end{tikzcd}$$

Here, the double-arrow is the degenerate 1-simplex $s_0 Y$, and this arrow has a direction to indicate the ordering of the vertices of the 2-simplex.

The intuition is that a 2-simplex like this in the homotopy coherent nerve (see next lecture) would represent a homotopy from $\mathrm{id}_Y \circ f$ to g.

(a) Suppose $\mathcal{C}$ is an ∞-category. Show that the existence of any one of the above types of 2-simplices is equivalent to the existence of all other types. (So for example an L exists if and only if an L' exists.)

(b) Suppose $\mathcal{C}$ is an ∞-category. Define two edges f, g (with the same domain and codomain) to be *homotopic* if one type of 2-simplex (and hence all types of 2-simplices) above exists. Prove this defines an equivalence relation on the set of edges from X to Y.

Hint: At this point, you only have three ways of producing simplices – by using degeneracy maps, by using face maps, and by using the horn-filling condition.

II.2 Composition

An ∞-category does not come with data that says "given two arrows f_{12} and f_{01}, here is how you *define* an arrow that deserves to be called $f_{12} \circ f_{01}$."

In other words, it doesn't define a preferred operation on the collection of arrows called composition.

However, it does give data one can interpret as "an arrow equipped with evidence that the arrow deserves to be called a composition." Namely, given f_{12} and f_{01}, the weak Kan condition tells us there exists at least one triangle filling the Λ^2_1 horn given by the f_{ij}.

Given any such triangle $T : \Delta^2 \to \mathcal{C}$, we should interpret $d_1 T$ as a candidate for a composition, and T as exhibiting the data witnessing $d_1 T$ as deserving of this title (of a composition).

(a) Let $\mathcal{C}$ be an ∞-category. Show that "composition is well-defined up to homotopy." More precisely, suppose that T and S are two 2-simplices filling a fixed horn $\Lambda^2_1 \to \mathcal{C}$. Show that $d_1 T$ and $d_1 S$ are homotopic (in the sense of Exercise II.1).

II.3 Associativity

(a) Convince yourself that a functor $\Delta^1 \times \Delta^1 \to \mathsf{Top}$ contains data that looks like a "homotopy coherent" commutative square. (You can also replace Top by any ∞-category, of course.)

Beware of a point of possible confusion: The diagonal 1-simplex of $\Delta^1 \times \Delta^1$ provides "more" data than one would naively expect in a homotopy coherent square diagram.

(b) Exhibit an injection of simplicial sets $\Delta^1 \times \Delta^1 \to \Delta^3$.

(c) Using the interpretation of composition from Exercise II.2, convince yourself that the weak Kan condition for Λ^3_i demonstrates that composition (of three morphisms) is associative up to homotopy.

II.4 Commentary

You should notice that all of your proofs above had, at their core, a choice you could make because a horn-filling condition guaranteed the existence of something. (This is not surprising – if your only hypothesis is an existence condition, all your proofs should make use of existence.)

As we are about to see, more sophisticated categorical arguments will require us to *remember* the particular choices we made in these existence proofs. In other words, if something exists (e.g., a homotopy) we don't want to throw it away (e.g., forget the homotopy and remember only that two things "are homotopic"). Combinatorially, this manifests in proofs where you want to fill many horns – to do so, you'll need to remember some faces and horns you've already created.

II.5 Equivalence/isomorphisms

A morphism $f : X \to Y$ in an ∞-category is called an *equivalence* or an *isomorphism* if there exists a morphism $g : Y \to X$, and two 2-simplices S and T such that

$$d_0 S = g \qquad d_1 S = s_0 X, \qquad d_2 S = f$$

and

$$d_0 T = f \qquad d_1 T = s_0 Y, \qquad d_2 T = g.$$

We call g an *inverse*, or *homotopy inverse* to f.

(a) Let us say that X and Y are *isomorphic* (or *equivalent*) if there exists an isomorphism from X to Y. Show that isomorphism is an equivalence relation.

(b) Fix f. Show that any two choices of inverse g are homotopic.

(c) Let us call g a *left* inverse to f if there exists a 2-simplex S as above. If f admits a left inverse g, and if g is an isomorphism, show f is an isomorphism.

Hint: These exercises have obvious counterparts in classical category theory. Thinking carefully about the proofs in the classical setting will give you insight into what kinds of horn-filling you want to perform. As an even more explicit hint, try to replace every equality you use in the classical setting with a homotopy (e.g., simplex) in an ∞-category. Finally, if this hint seems unhelpful because you are not comfortable with classical category theory, pretend that X and Y are groups, and that f and g are group homomorphisms.

II.6 Left and right horn-filling

(a) Suppose $\mathcal{C}$ is a Kan complex (and in particular, an ∞-category). Show that every morphism in $\mathcal{C}$ is an isomorphism.

(b) Suppose $\mathcal{C}$ is a simplicial set satisfying the horn-filling condition for all n and for all $0 \leq i < n$. (In particular, $\mathcal{C}$ is an ∞-category.) Show that every morphism in $\mathcal{C}$ is an isomorphism.

In fact, any ∞-category for which every morphism is an isomorphism is a Kan complex. Proving this by hand is too involved; there are slick tools using fibration-style arguments that we'll see later.

II.7 Opposites

(a) Exhibit an isomorphism of the category Δ to itself that is the identity on objects, but "reverses the order of the morphisms." (Motivation: Every category has an opposite. Study the nerve $N(\mathcal{C})$ of a category $\mathcal{C}$ and $N(\mathcal{C}^{op})$ of the opposite category; how are the face and degeneracy maps are exchanged?)

Remark II.7.0.1. You can now tackle a version of Exercise II.6 while assuming the horn-filling condition for $0 < i \leq n$.

Lecture III

Examples of infinity-categories and functors

In this lecture we defined the infinity-category of topological spaces (and more generally, the homotopy coherent nerve of a topologically enriched category). We worked out the 0-, 1-, 2-, and 3-simplices in the homotopy coherent nerve of topological spaces explicitly. (Note: the $n = 3$ case uses the notation $\Delta^{1,1}$, but this should be $\Delta^1 \times \Delta^1$.)

We also saw how to turn any dg-category into an infinity-category using the dg nerve of Lurie.

Finally, we saw basic examples of functors of infinity-categories. We saw how naturally functors can encode homotopy-commuting squares, homotopy-coherent actions of groups, and local systems.

We cautioned that though a functor $\Delta^1 \times \Delta^1 \to \mathsf{Top}$ should be thought of as a homotopy-commuting square, a functor of this form is not literally a homotopy-commuting square. (See Example 1 of functors in the notes. In the notation there, the homotopy $f_{12} \circ f_{01} \sim f'_{12} \circ f'_{01}$ results from a choice of composite of the homotopies H and H'.)

There was one technical point about how to define an ∞-category that "correctly" captures the homotopy theory of chain complexes over an arbitrary base ring R (as opposed to, say, when R is a field). The usual homological cautions – taking injective or projective chain complexes – apply here. I did not fixate on this issue, as it was a bit tangential to my desire to show that the combinatorial language of simplicial sets encodes homotopy-coherent ideas with great precision.

We recommend Chapter 1 of Higher Topos Theory [8] and Section 1.3 of

Higher Algebra [9] for further reading.

Note: I wanted to talk about (right) fibrations in this lecture, but this was postponed to the next lecture.

2023-8-2 (水) HA in GT ③

(Last time : ∞-cat.s + examples

Today : More examples,
(Right) fibrations)

Recall

- A simp. set is a functor

$$\Delta^{op} \longrightarrow \text{Sets}$$

$$\left(\rightsquigarrow \quad X_0 \leftleftarrows X_1 \; X_2 \; X_3 \cdots \right)$$

- An ∞-category (resp. Kan cpx.) is a simp. set $\mathcal{C}$

satisfying :

$0 < {}^{\forall} i < {}^{\forall} n$,
(resp. $0 \leq {}^{\forall} i \leq {}^{\forall} n$)

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

E.g. $\mathcal{T}op$: the ∞-cat. of spaces.

$$\mathcal{T}op_0 := \{ \text{spaces} \simeq^{\exists} \text{CW cpx.s} \}$$
$$\upuparrows \downarrow$$
$$\mathcal{T}op_1 := \{ f : X \to Y : \text{cont.} \}$$
$$\Uparrow \Downarrow$$
$$\mathcal{T}op_2 := \{ (X_0, X_1, X_2 ; f_{01}, f_{12}, f_{02} ; f_{02} \xrightarrow{H} f_{12} \circ f_{01}) \}$$
$$\vdots$$

$$\begin{array}{ccc} Y & & X \\ {\scriptstyle d_0}\uparrow & & \uparrow{\scriptstyle d_1} \\ & f & \end{array} \qquad \begin{array}{c} X \\ \downarrow{\scriptstyle s_0} \\ \mathrm{id}_X \end{array}$$

?

As its preparation ...

Notation Fix $n \geq 0$, $i \leq j \in [n]$.

$$\boxed{P_{i,j} := \{ I \subset [n] \mid \min I = i,\ \max I = j \}.}$$

the power set of the interval (i,j) ! ($\rightsquigarrow$ poset!)

e.g.

$$P_{2,2} = \{ \{2\} \}.$$

$$P_{4,6} = \left\{ \begin{array}{c} \{4,6\}, \\ \cap \\ \{4,5,6\} \end{array} \right\}$$

$$P_{0,3} = \left\{ \begin{array}{ccc} & \{0,3\} & \\ \cap & & \cap \\ \{0,1,3\} & & \{0,2,3\} \\ \cap & & \cap \\ & \{0,1,2,3\} & \end{array} \right\}$$

$\rightsquigarrow N(P_{i,j})$ is a simp. set $\left(\cong (\Delta^1)^{j-i-1} \right)$.

We have

$$\begin{array}{ccc} P_{j,k} \times P_{i,j} & \longrightarrow & P_{i,k} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (I', I) & \longmapsto & I \cup I' \end{array}$$

$$\rightsquigarrow N(P_{j,k}) \times N(P_{i,j}) \to N(P_{i,k}).$$

<u>E.g.</u> (continued)

An element of $\mathcal{T}op_n$ is defined to be data of

- Top spaces $X_1, \dots, X_n$;
- $0 \le \forall i \le \forall j \le n$ A map of simp. sets

$$N(P_{i,j}) \xrightarrow{\alpha} \mathrm{Sing}(\underbrace{\hom(X_i, X_j)}_{= C^0(X_i, X_j)})$$

satisfying :

①
$$\begin{tikzcd} N(P_{j,k}) \times N(P_{i,j}) \arrow[rr] \arrow[d, "\alpha \times \alpha"] & & N(P_{i,k}) \arrow[d, "\alpha"] \\ \mathrm{Sing}(\hom(X_j, X_k)) \times \mathrm{Sing}(\hom(X_i, X_j)) \arrow[rr] \arrow[urr, phantom, "\circlearrowright"] & & \mathrm{Sing}(\hom(X_i, X_k)) \end{tikzcd} ;$$

②
$$\begin{array}{ccc} N(P_{i,i}) & \xrightarrow{\alpha} & \mathrm{Sing}(\hom(X_i, X_i)) \\ \| & & \rotatebox{90}{$\in$} \\ \{\bullet\} \ni \bullet & \longmapsto & \mathrm{id}_{X_i} \end{array}.$$

$n=0$

$$\begin{array}{ccc} N(P_{0,0}) & \longrightarrow & \mathrm{Sing}(\hom(X_0, X_0)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bullet & \longmapsto & \mathrm{id}_{X_0} \end{array}$$

n=1

$$\begin{array}{ccc} N(P_{0,1}) & \to & \mathrm{Sing}(\hom(X_0,X_1)) \\ \| & & \text{\rotatebox{90}{$\in$}} \\ \Delta^0 \ni 0 & \longmapsto & f_{01} \end{array}$$

cont. map. を1個指定するだけ.

n=2

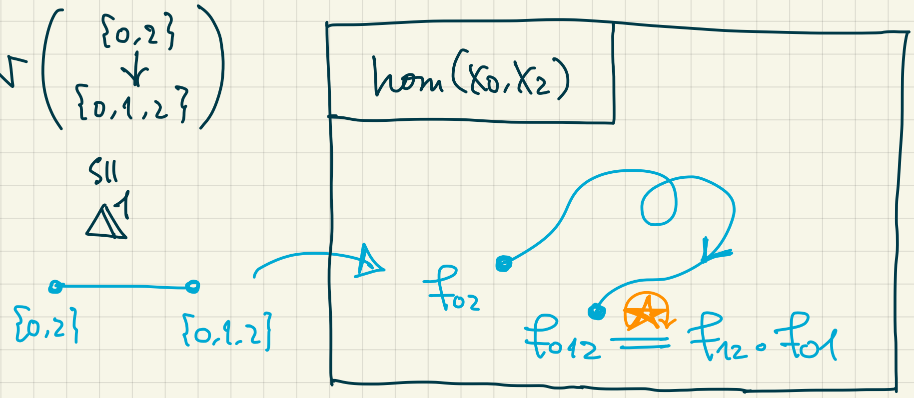
$$N(P_{0,2}) \to \mathrm{Sing}(\hom(X_0,X_2))$$

$$\| $$

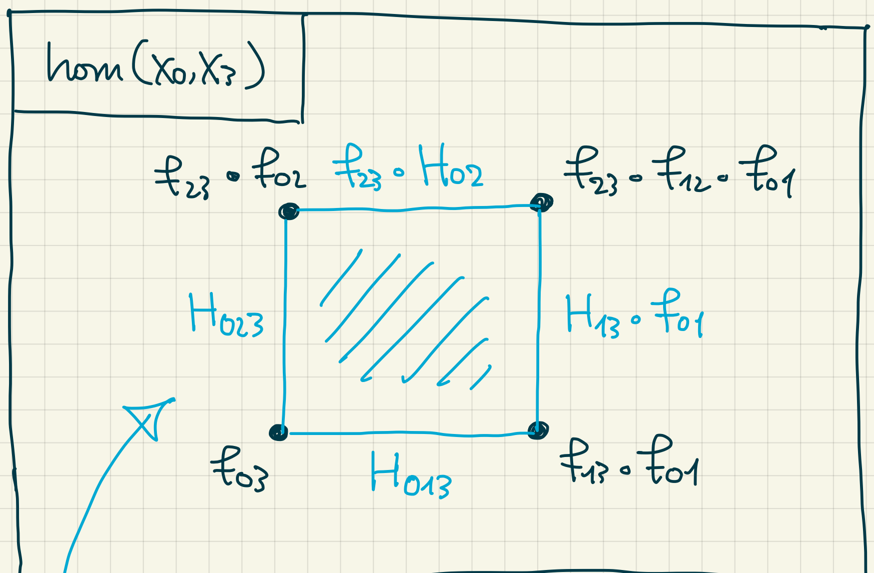
$$N\left(\begin{array}{c}\{0,2\} \\ \downarrow \\ \{0,1,2\}\end{array}\right)$$

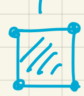
$$\cong$$

$$\Delta^1$$



n=3





$$\left(\because \; N(P_{0,3}) = \left(\begin{tikzcd}[column sep=small] & \{0,3\} \arrow[dl] \arrow[dr] & \\ \{0,1,3\} \arrow[dr] & & \{0,2,3\} \arrow[dl] \\ & \{0,1,2,3\} & \end{tikzcd} \right) \cong \Delta^{1,1}. \right)$$

E.g. Let $\mathcal{D}$ be a cat.
s.t. homs are spaces & "$\circ$" is cont.

composition

The homotopy coherent nerve $N(\mathcal{D})$ is ⋯

An element of $N(\mathcal{D})_n$ is defined to be data of :

- objects $X_0, \ldots, X_n$;
- $0 \leq \forall i \leq \forall j \leq n$ A map of simp. sets

$$N(P_{i,j}) \xrightarrow{\alpha} \mathrm{Sing}(\mathrm{hom}(X_i, X_j))$$

space !

satisfying the above ① & ② .

Thm.

> For any Top-enriched $\mathcal{D}$,
> $N(\mathcal{D})$ is an ∞-category.

E.g. Fix a dg-cat. $\mathcal{A}$.

$\forall X_0, X_1 \in \mathrm{Obj}(\mathcal{A})$.

- $\hom(X_0, X_1)$ is a chain cpx. ;
- Composition

$$\hom(X_1, X_2) \otimes \hom(X_0, X_1) \to \hom(X_0, X_2)$$

satisfies Leibniz, and is unital and associative.

Def. (Lurie)

$N_{dg}(\mathcal{A})$ is the simp. set whose n-spx.s are :

If $\mathcal{A}$ is the dg-cat. of chain complexes ...

$\underline{n=0}$: objs of $\mathcal{A}$;

$\underline{n=1}$: $(X_0, X_1, f \in \hom(X_0, X_1))$ s.t. $|f| = 0$ & $df = 0$;

chain map !

$\underline{n=2}$: $(X_0, X_1, X_2 ; f_{01}, f_{02}, f_{12} ; H)$

s.t. $H \in \hom^{-1}(X_0, X_2)$

$dH = f_{02} - f_{12} \circ f_{01}$.

chain htpy !

$\underline{\text{general}}$: data of ...

$\forall I \subset [n]$ w/ $\# I \geq 2$,

$f_I \in \hom^{2-\#I}(X_{\min I}, X_{\max I})$

s.t. (i) each $\{f_I\}_{I \subset J}$ is a J-spx. ;

(ii) $$df_{[n]} = \sum_{0<i<n} (-1)^{i-1} f_{[n]\setminus\{i\}} + \sum (-1)^i f_{\geq i} \circ f_{\leq i}.$$

Thm.

For any dg-cat. $\mathcal{A}$,
$N_{dg}(\mathcal{A})$ is an ∞-category.

Examples of functors:

① $\Delta^1 \times \Delta^1 \to \mathcal{T}op$

$$
\begin{tikzcd}
X_0 \arrow[r, "f_{01}"] \arrow[d, "f'_{01}"'] \arrow[dr, "f_{02}" description] & X_1 \arrow[d, "f_{12}"] \arrow[dl, phantom, "H" near start] \\
X'_1 \arrow[r, "f'_{12}"'] \arrow[ur, phantom, "H'" near start] & X_2
\end{tikzcd}
$$

$\boxed{\hom(X_0, X_2)}$

$$
\begin{tikzcd}
f_{02} \arrow[r, "H"] \arrow[d, "H'"'] & f_{12} \circ f_{01} \\
f'_{12} \circ f'_{01} &
\end{tikzcd}
$$

We can fill up this!

(☺ Kan cpx.!)

This functor is a htpy commuting square.

② $\Delta^2 \to N_{dg}(\mathrm{Chain})$

$=$ a 2-spx. in $N_{dg}(\mathrm{Chain})$

$$
= \quad
\begin{tikzcd}
X_0 \arrow[r, "f_{01}"'] \arrow[rr, bend left, "f_{02}"] & X_1 \arrow[r, "f_{12}"'] \arrow[u, phantom, "\Updownarrow H"] & X_2
\end{tikzcd}
.
$$

③ Fix a (discrete) group G.

Think of G as a cat. BG w/ one object.

Consider $N(BG)$

$$\begin{array}{c} N(BG) \\ \| \end{array}$$

$$\begin{array}{ccccc} & & g_{12}g_{01} \xleftarrow{\ d_0\ } & & \\ & & g_{12} \xleftarrow{\ d_1\ } & (g_{01}, g_{12}) & \\ & & g_{01} \xleftarrow{\ d_2\ } & \cap & \\ & & \cap & & \end{array}$$

$$\left\{ \begin{array}{ccccccc} * & \Leftarrow & G & \Lleftarrow & G \times G & \Lleftarrow & G^{\times 3} \cdots \\ 0 & & 1 & & 2 & & 3 \end{array} \right\}$$

⇝ What is a functor $BG \to \mathrm{Top}$?

0 : $\bullet \longmapsto X$

1 : $g \longmapsto \rho_g : X \to X$

2 : $\begin{tikzcd} & {} \\ {} \arrow[ur, "g_{02}"] \arrow[r, "g_{01}"'] & {} \arrow[u, "g_{12}"'] \end{tikzcd} \longmapsto$ A htpy $\rho_{g_{02}} \sim \rho_{g_{12}} \circ \rho_{g_{01}}$.

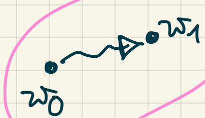
"group action $G \curvearrowright X$ up to htpy"

⋮

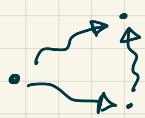
④ Fix a (reasonable) space W.

What is a functor $\mathrm{Sing}(W) \to N_{dg}(\mathrm{Chain})$?

0 : $w \longmapsto V_w$

1 : 

$\longmapsto$ $V_{w_0} \xrightarrow{f} V_{w_1}$: chain map

2 :  $\longmapsto$ a htpy H s.t.
$dH = f_{02} - f_{12} \circ f_{01}$

$\exists$ inverse path! ($\because$ $\mathrm{Sing}(W)$ is a Kan cpx.)

→ Each f is a chain htpy equiv!

Upshot:

A local system (of chain cpx.s) on W is a functor $\mathrm{Sing}(W) \to N_{dg}(\mathrm{Chain})$.

Def.

Fix a simp. set K and an ∞-cat. $\mathcal{C}$.

Define a simp. set $\mathrm{Fun}(K, \mathcal{C})$ as follows:

$$\mathrm{Fun}(K, \mathcal{C})_n := \{ \Delta^n \times K \xrightarrow{f} \mathcal{C} \}.$$

Thm. This is an ∞-cat. $\circ\circ$ (called the ∞-cat. of functors from K to $\mathcal{C}$.)

Def.

The cat. of local systems on W is

$$\mathrm{Fun}(\mathrm{Sing}(W), N_{dg}(\underline{\mathrm{Chain}})).$$

(Prof. Hiro's Comment)

※ This should be the dg-cat. of chain cpxes over some field.
Otherwise, we should restrict to the dg-cat. of inj., or proj., chain cpxes over your base ring.

Exercises: Examples of ∞-categories

III.1 dg-nerves

Fix a dg-category $\mathcal{A}$ (say, over the integers).

(a) Write out explicitly all the data encoding a 3-simplex in the dg-nerve of $\mathcal{A}$.

(b) Verify that the dg-nerve of $\mathcal{A}$ is an ∞-category.

(c) Show that a functor between dg-categories induces a functor between the nerves of the dg-category.

III.2 Homotopy coherent nerve

Fix a category $\mathcal{C}$ enriched over Kan complexes. (This means for all $x, y \in \operatorname{Ob} \mathcal{C}$, $\hom(x, y)$ is a Kan complex, and composition is a map $\hom(y, z) \times \hom(x, y) \to \hom(x, z)$ of simplicial sets.)

We define the homotopy coherent nerve $N(\mathcal{C})$ of $\mathcal{C}$ identically to that for a topologically enriched category (skipping all instances of Sing.)

It is a theorem that $N(\mathcal{C})$ is an ∞-category.

(a) Verify that all inner 2-horns in $N(\mathcal{C})$ can be filled.

(b) Verify that a functor between Kan-complex enriched categories induces a functor between their nerves.

III.3 The Kan complex $\mathcal{C}^{\simeq}$

Fix an ∞-category $\mathcal{C}$.

Define $\mathcal{C}^{\simeq} \subset \mathcal{C}$ to be the simplicial set whose k-simplices are defined as follows:

- $(n = 0)$. $\mathcal{C}^{\simeq}_0 = \mathcal{C}_0$.
- $(n = 1)$. $\mathcal{C}^{\simeq}_1$ consists of those edges of $\mathcal{C}$ that are isomorphisms (Exercise II.5).
- For all larger n, $\mathcal{C}^{\simeq}_n$ consists of those n-simplices of $\mathcal{C}$ all of whose edges are isomorphisms.

$\mathcal{C}^{\simeq}$ is a Kan complex. It is in fact the largest Kan complex contained inside $\mathcal{C}$.

(a) Show that $(\mathcal{C} \times \mathcal{D})^{\simeq} \cong \mathcal{C}^{\simeq} \times \mathcal{D}^{\simeq}$.

(b) Fix a Kan complex K. Exhibit a natural isomorphism

$$\hom(K, \mathcal{C}) \xleftarrow{\cong} \hom(K, \mathcal{C}^{\simeq}) \tag{III.3.1}$$

where hom denotes the set of simplicial set maps. (Here, naturality means that the above isomorphism commutes with post-composition by maps $f : \mathcal{C} \to \mathcal{C}'$ and with pre-composition by maps $K \to K'$.)

(c) Show that the inclusion of the category of Kan complexes into the category of ∞-categories is a fully faithful left adjoint.

(d) Note that the righthand side of (III.3.1) is the set of 0-simplices of the simplicial set $\mathsf{Fun}(K, \mathcal{C}^{\simeq})$ (as introduced in this lecture). How would you define a simplicial set on the lefthand side to enrich (III.3.1) into an isomorphism of *simplicial sets* from $\mathsf{Fun}(K, \mathcal{C}^{\simeq})$? How does your answer compare to $\mathsf{Fun}(K, \mathcal{C})^{\simeq}$?

III.4 The ∞-category of ∞-categories

(a) Fix two ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Verify that $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is a simplicial set (by declaring its k-simplices to consists of simplicial set maps $\Delta^k \times \mathcal{C} \to \mathcal{D}$).

(b) Define a composition map $\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \times \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

(c) Assume that $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is an ∞-category whenever $\mathcal{D}$ is an ∞-category. (This is a theorem.) Show that your composition map induces maps $\mathsf{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \times \mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \to \mathsf{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$. (See Exercise III.3.) In particular, show that the collection of ∞-category forms a Kan-complex enriched category.

By applying the homotopy coherent nerve to the Kan-complex enriched category of ∞-categories, we obtain an ∞-category we denote by

$$\mathcal{C}\mathrm{at}_{\infty},$$

and which we call the ∞-category of ∞-categories.

Lecture IV

Fibrations

We began this lecture with a tangential analogy. In Japan, it is common for a student of kyudo (archery) to not touch an arrow for years. Instead, they train their body in the proper form of readying a bow. Then, all one needs is to be handed an arrow to start developing one's aim and piercing targets.

The classical theory of categories is the bow. Indeed, mathematicians have trained for years developing our category-theoretic muscles. The combinatorial model of infinity-categories is the arrow, the thing that actually allows us to pierce our prey. I have heard many complaints that the refrain "infinity-categories allow you to use categorical arguments as though they work just fine in homotopical settings" (a truism that sounds too good to be true) is frustrating for its appeal and its lack of detail; perhaps it becomes slightly less frustrating if framed with the current analogy.

I emphasized that in these lectures, I focus on the arrow, not the bow. This is because there are plenty of examples and references on the use of the bow. (For example, Mac Lane's classic [11].)

I then delved into the actual content of the lecture: the theory of fibrations, which I unfortunately rushed through for lack of time. The two large take-aways are as follows.

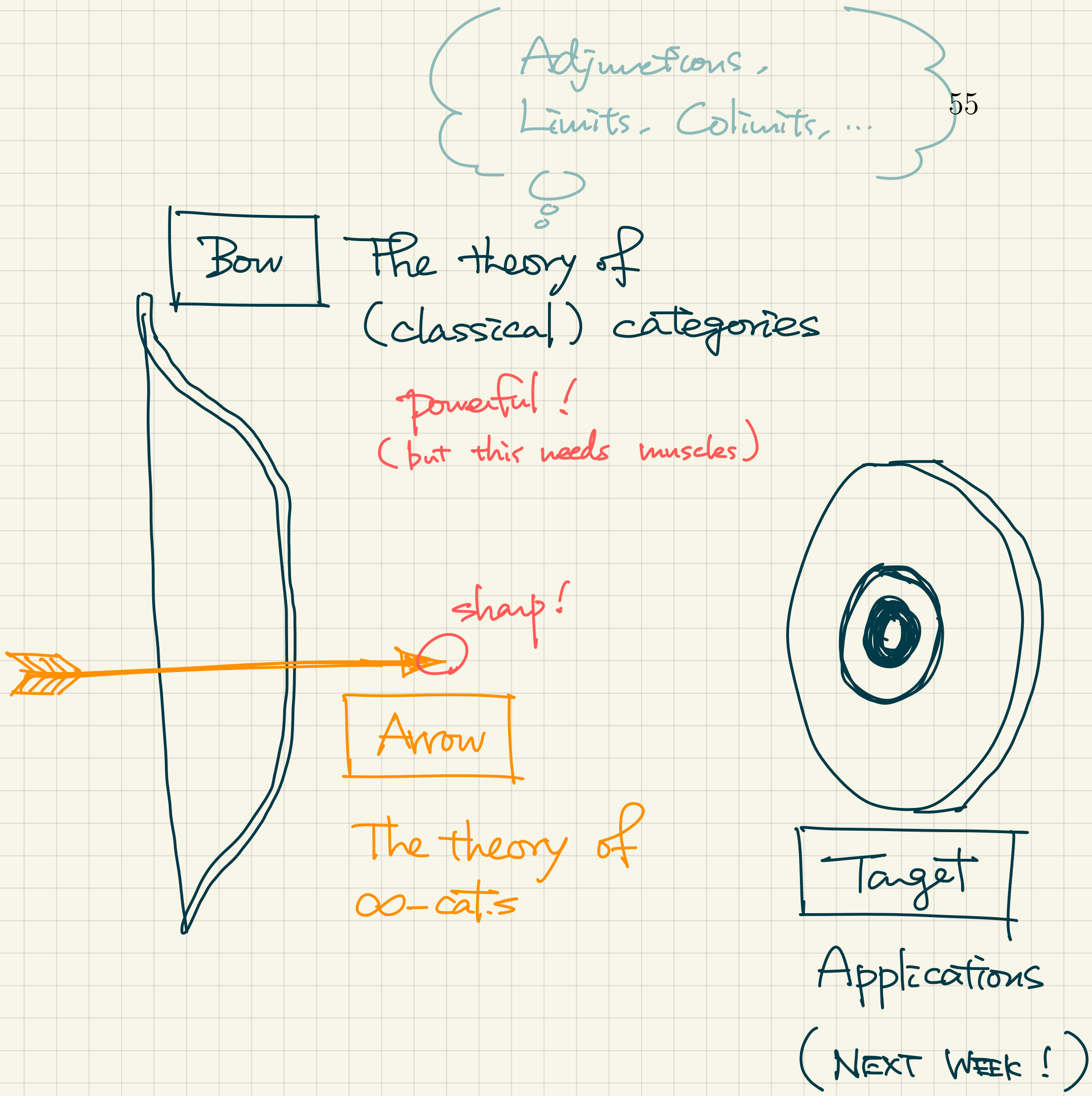
(I) Fibrations over B encode certain functors from B (covariant or contravariant; to either the infinity-category of spaces or the infinity-category of infinity-categories; depending on the type of fibration). Fibrations are amazing because often, constructing a functor is far harder than constructing a fibration – this is familiar to algebraic geometers who define stacks as categories fibered in groupoids. The motivating example was the category of principle G-bundles (as a right fibration over the category of spaces).

I remarked that one is tempted to define the assignment of a space X to the groupoid $Bun_G(X)$ as a contravariant functor, but that in fact composition is not respected on the nose unless one makes some arbitrary choices. This example shows both the utility of fibrations 1-categorically, and the necessity of higher-categorical notions to encode something as classical as "the way in which composition is respected by pullback."

(II) Fibrations are defined as maps having lifting properties with respect to a simple class of morphisms; but in the course of life one must often ask if fibrations have lifting properties with respect to larger classes of morphisms. I rushed through the idea of the small object argument, which allows us to prove powerful theorems about fibrations by identifying which morphisms we can lift. Examples were mostly left to the exercises.

Amazingly, much of the theory of infinity-categories can be established by utilizing concrete combinatorics (which we practiced in preceding exercises) and by small-object-argument-esque techniques (which we see in this lecture's exercises).

For further reading regarding the small object argument, I recommend A.1.2 of Higher Topos Theory and Section 1.4.4 of Kerodon. (Sections 1.4.5 and 1.4.6 of Kerodon also give applications of the kinds of arguments one uses to study lifting properties in simplicial sets.) Section 4.2 of Kerodon also gives a nice introduction to left and right fibrations.


Adjunctions,
Limits, Colimits, ...
Bow
The theory of
(classical) categories
powerful!
(but this needs muscles)
sharp!
Arrow
The theory of
∞-cats
Target
Applications
(NEXT WEEK!)

2023-8-3 (木) HA in G ④

Last times : ∞-cat.s + examples
↑ combinatorics (of simplices)

Today : (Right) fibrations
↑ Quillen's small object argument from the theory of model cat.s

Def. Fix a map $\begin{array}{c} E \\ p\downarrow \\ S \end{array}$ of simp. sets.

p is called an inner fibration if :

$0 < {}^\forall i < {}^\forall n$,
$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall F"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "\forall G"'] \arrow[ur, dotted, "\exists"] & S . \end{tikzcd}$$

E.g. If $S = \Delta^0$, then

p is an inner fib. $\overset{!}{\Leftrightarrow}$ E is an ∞-cat.

Rmk. If p is an inn. fib., then

${}^\forall s_0 \in S_0$, $p^{-1}(s_0)$ is an inn.

$$(p^{-1}(s_0))_n = (E \times_S \Delta^0)_n \quad \because \begin{tikzcd} p^{-1}(s_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \Delta^0 \arrow[r, "s_0"] & S \end{tikzcd}$$

$$= \{ (A, B) \in \Delta^0_n \times E_n \mid s_0(A) = p(B) \}$$

$$\cong \{ B \in E_n \mid p(B) = s_0 \}.$$

$(= s_0 \cdots s_0(s_0))$

i.e. $p(B)$ is the degen. n-spx. at s_0.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall F"] \arrow[d] & p^{-1}(s_0) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] \arrow[urr, dashed, "\exists"'] & \Delta^0 \arrow[r] & S \end{tikzcd}$$

(p is an inn. fib.)

(commutativity of $\nearrow\downarrow$, $\rightarrow\rightarrow$)

//

Def. (continued)

- p is a **left fibration** (resp. right / Kan) if we can take $0 \leq i < n$. (resp. $0 < i \leq n$ / $0 \leq i \leq n$).

- p is a **trivial (Kan) fibration** if $\forall n$,

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\forall F"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "\forall G"'] \arrow[ur, dashed, "\exists"] & S. \end{tikzcd}$$

$\partial\Delta^n : \Delta^{op} \to \mathrm{Sets}$ is defined as follows :

$$[m] \longmapsto \left\{ f : [m] \to [n] \;\middle|\; \begin{array}{l} f : \text{weakly order pres.} \\ \& \text{ NOT surj.} \end{array} \right\}$$

E.g. Fix a top. group G.

Consider the category Bun_G, where

- $\mathrm{Obj}(\mathrm{Bun}_G) = \left\{ \begin{array}{c} P \\ \downarrow \\ X \end{array} : \text{principal } G\text{-bdles} \right\}$.
- $\hom_{\mathrm{Bun}_G}\left(\begin{array}{c} P \\ \downarrow \\ X \end{array}, \begin{array}{c} P' \\ \downarrow \\ X' \end{array} \right)$

$$= \left\{ (f, \alpha) \;\middle|\; \begin{array}{l} f : X \to X' : \text{cont.}, \\ \alpha : f^* P' \to P : \text{isom.} \end{array} \right\}.$$

$$\rightsquigarrow \quad \begin{array}{c} \mathrm{Bun}_G \\ \downarrow p \\ \mathrm{Top} \end{array} \qquad \begin{array}{c} \boxed{\begin{array}{c} P \\ \downarrow \\ X \end{array}} \\ \mapsto \\ \boxed{X} \end{array}$$

$\to$ **Prop.** $\begin{array}{c} N(\mathrm{Bun}_G) \\ \downarrow \\ N(\mathrm{Top}) \end{array}$ is a right fib.

...

e.g. $(n = 1 = \tilde{n})$

$$\begin{tikzcd} P_1 \arrow[d] \\ X_1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[d, hook] & \Lambda^1_1 \arrow[r, "F"] \arrow[d] & N(\mathrm{Bun}_G) \arrow[d] \\ 0 \arrow[r] & 1 \quad \Delta^1 \arrow[r, "G"'] \arrow[ur, dashed, "\exists \tilde{G}?"] & N(\mathrm{Top}) \end{tikzcd}$$

$$\boxed{X_0} \xrightarrow{f} \boxed{X_1}$$

$$\rightsquigarrow \tilde{G}(\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet})$$

$$:= \left(\begin{tikzcd} f^*P_1 \arrow[r, "\overset{\alpha}{\cong}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_1 \arrow[d] \\ X_0 \arrow[r, "f"] & X_1 \end{tikzcd} \right)$$

How to remember

Iran-Contra

※ Right fib.s encode contravariance ;
(pull-back!)

Left ——"—— covariance.
(push-forward!)

Moreover, note that

$(\mathrm{Bun}_G)_X = p^{-1}(X)$ has all invertible morphisms.

Fact

If $p : E \to S$ is a left/right fib., then $\forall s \in S_0$, $p^{-1}(s)$ is a Kan cpx.

In other words, if $S = \Delta^0$, then TFAE:

- p : left fib. ;
- p : right fib. ;
- p : Kan fib.

Fact

The theory of right fib.s over S left

is equivalent to the theory of functors

$$S^{op} \longrightarrow \mathcal{T}op.$$

S

For Bun_G, we have,

$\forall f : S \to T$ in Top,

a functor $(\mathrm{Bun}_G)_S \xleftarrow{f^*} (\mathrm{Bun}_G)_T$.

Rmk. We've seen

$$\begin{array}{ccc} \mathrm{Top} & \longrightarrow & \text{Kan cpx.s} \\ \in & & \in \\ W & \longmapsto & \mathrm{Sing}(W) \end{array}.$$

We also have

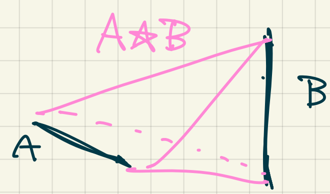
$$\text{Kan cpx.s} \xrightarrow{\ (!)\ } \mathrm{Top}$$

$$X \longmapsto |X| := \left(\coprod_{n \geq 0} X_n \times \Delta^n_{\mathrm{top.}} \right) \Big/ \sim$$

the usual "topological" n-spx.

the geometric realization of X.

胞体分割の構成!



Interlude : Joins

Fix two simp. sets A and B.

$$\rightsquigarrow (A \star B)_n := \left\{ (\pi, f_A, f_B) \;\middle|\; \begin{array}{l} \pi : \Delta^n \to \Delta^1 \\ \pi^{-1}(0) \xrightarrow{f_A} A \\ \pi^{-1}(1) \xrightarrow{f_B} B \end{array} \right\}$$

the join from A to B

Convention : $A_{-1} = B_{-1} = \mathrm{pt.}$

$$\cong \coprod_{\substack{-1 \leq i \leq n \\ -1 \leq j \leq n \\ i+j = n-1}} A_i \times B_j .$$

Def. Fix $\mathcal{C}$ an ∞-cat. and $x \in \mathcal{C}_0$.

Define the slice ∞-category $\mathcal{C}_{/x_0}$ as follows:

An n-spx. is a map $\Delta^n \star \Delta^0 \to \mathcal{C}$ $\left(\overset{\cong}{\Delta^{n+1}}\right)$ s.t. $f|_{\Delta^0} = x$.

e.g.

(n=0) $\Delta^1 \xrightarrow{f} \mathcal{C}$ s.t. $f(1) = x$.

$$0 \bullet \longrightarrow \bullet 1 \quad \mapsto \quad y_0 \longrightarrow x$$

(n=1) $\Delta^2 \xrightarrow{f} \mathcal{C}$ s.t. $f(2) = x$.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ & & 2 \\ 0 \arrow[uu] \arrow[urr] & & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} y_1 \arrow[r] & x \\ y_0 \arrow[u] \arrow[ur] & \end{tikzcd}$$

$\vdots$

$\rightsquigarrow$ We have a map

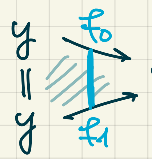
$$\begin{array}{ccc} \mathcal{C}_{/x} & & \Delta^n \star \Delta^0 \xrightarrow{f} \mathcal{C} \\ \downarrow p & & \downarrow \\ \mathcal{C} & & f|_{\Delta^n}. \end{array}$$

<u>Prop.</u> This is a right fibration.

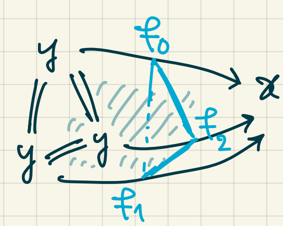
? For $y \in \mathcal{C}$, what is the fiber $p^{-1}(y)$?

$p^{-1}(y)_n$ is ...

(n=0) $\{ y \longrightarrow x \}$

(n=1) $\Bigg\{$ 

$\Bigg\}$ $\leftarrow$ htpy !

(n=2) $\Bigg\{$ 

$\Bigg\}$

Thus,

$\rightsquigarrow$ The simp. set $p^{-1}(y)$ is
the space of morphisms from y to x;
it captures all maps, htpys between them, etc.

Back to trivial fibrations :

Prop. If p is a triv. fib., then

$$\begin{tikzcd} {}^{\forall}A_0 \arrow[r, "\forall"] \arrow[d, hook, "\forall\,\text{inj.}"'] & E \arrow[d, "p"] \\ {}^{\forall}A \arrow[r, "\forall"'] \arrow[ur, dotted, "\exists"] & S \end{tikzcd}.$$

Often, we want to prove that p is a "BLAH" fibrations.
The proof often comes down NOT to lifts involving the original BLAH-defining maps, but other maps.

Thm. (Small Object Argument)

The class of maps $\{A_0 \to A\}$ that allow triv. fib. to admit lifts is the class of injections.

The conv. holds!

Def.

A map $A_0 \to A$ is called inner (resp. left / right) anodyne if it admits a lifting solution for $\forall$ inner (resp. left / right) fib.s.

⊛ "Anodyne" = "harmless," or "relieves pain."

(Sketch of Prop.)

Let T be the collection $\left\{ \begin{smallmatrix} A_0 \\ \downarrow \\ A \end{smallmatrix} \right\}$ of mor.s that admit lifts.

Claim ① T is closed under

- pushouts,
- (transfinite) compositions,
- retracts.

Claim ② $\forall A_0$, the incl. $\ulcorner A_0 \hookrightarrow A_0 \sqcup_{\partial\Delta^k} \Delta^k \lrcorner \in T$.

This is a consequence of Claim ① and the hypothesis that T includes the boundary-inclusions of simplices.

∎

Exercises: Joins and fibrations

IV.1 Join preserves colimits in each variable

Fix a simplicial set A.

(a) Suppose one has a diagram $\{B_d\}_{d\in D}$ of simplicial sets – i.e., a functor from some category D to the category of simplicial sets. Show that

$$\mathrm{colim}_{d\in D}(A \star B_d) \to A \star (\mathrm{colim}_{d\in D} B_d)$$

is an isomorphism of simplicial sets.

If you are not used to the language of colimits, here are two examples you can try out. I will make explicit one way you can spell out the desired isomorphism.

(i) Pushouts: Suppose $B = B_1 \bigcup_{B_0} B_2$ as a simplicial set. Concretely, I mean that there exists a commutative diagram of simplicial sets

$$\begin{tikzcd} B_0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B \end{tikzcd}$$

and that for every $k \geq 0$, the map between the sets of k-simplices

$$(B_1)_k \bigcup_{(B_0)_k} (B_2)_k \to B_k$$

is a bijection. (Here, the union is simply the quotient set of $(B_1)_k \coprod (B_2)_k$ obtained by identifying the common elements coming from $(B_0)_k$.)

Then show that there is an induced map

$$(A \star B_1) \bigcup_{A \star B_0} (A \star B_2) \to A \star B$$

and that this map is an isomorphism of simplicial sets. (Equivalently, show that for every k, the function between the sets of k-simplices is a bijection.)

(ii) Filtered colimits: Suppose you have a sequence of simplicial sets $B_0 \to B_1 \to \ldots$ (you can take the indexing set to be a bigger ordinal than $\mathbb{N}$ if you like) and set $B = \bigcup_i B_i$. This is the simplicial set whose k-simplices are obtained by quotienting the set $\coprod_k (B_i)_k$ by identifying elements that arise from common elements in the sequence of maps between simplicial sets $B_i \to B_j$.

Show that there is an induced map $\bigcup_i (A \star B_i) \to A \star (\bigcup_i B_i)$ and that this map is an isomorphism.

IV.2 Some closure properties of fibrations

Remark IV.2.0.1. Note that the solutions to this exercise should have a distinct flavor from the solutions to previous exercises – they should involve no combinatorial arguments particular to simplicial sets. (The particulars of simplicial sets only arises if you must convince yourself that pullbacks exist, and that pullbacks of simplicial sets behave as dimension-wise pullbacks of sets.) The solutions to these exercises, instead, should give you practice with manipulating commutative diagrams.

(a) Show that the composition of two inner fibrations is an inner fibration.

(b) Let $p : E \to S$ be an inner fibration and $f : S' \to S$ a map of simplicial sets. Show that the pullback of p along f is also an inner fibration.

(c) Convince yourself that the previous two exericses are true if you replace inner fribations with

  (i) Left fibrations
  (ii) Right fibrations
  (iii) Kan fibrations
  (iv) Trivial fibrations.

IV.3 The over-category of an object

Let $\mathcal{C}$ be an ∞-category and fix an object x.

(a) Verify that the forgetful functor $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration.

(b) Prove that $\mathcal{C}_{/x}$ is an ∞-category. (Hint: Exercise IV.2.)

IV.4 Closure of left lifts

Remark IV.4.0.1. You may take every map in this problem to be a map of simplicial sets, but this exercise can be completed in any category.

However, you will want to at least know about the *universal properties* of colimits. This is a classical topic in the theory of (ordinary, non-infinity) categories, and will be useful even when you restrict this exercise to the setting of simplicial sets (rather than an arbitrary category).

The reason you'll want to know about the universal properties is that these universal properties allow you to conclude the existence of certain arrows that render certain diagrams commutative. This flavor of argument will be useful in these exercises.

Let F be a class of morphisms, and fix an element $p : E \to S$ in F. We say that a morphism $j : A_0 \to A$ has the *left lifting property* with respect to p if

$$\begin{tikzcd} A_0 \arrow[r, "\forall F"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ A \arrow[r, "\forall G"] \arrow[ur, dashed, "\exists"] & S. \end{tikzcd}$$

We let $L(F)$ denote the collection of morphisms j for which j has the left lifting property with respect to all elements of F.

(a) Show that $L(F)$ is closed under pushouts. That is, given a pushout square

$$\begin{tikzcd} A_0 \arrow[r, "j"] \arrow[d] & A \arrow[d] \\ A_0' \arrow[r, "j'"] & A' \end{tikzcd}$$

if j is in $L(F)$, then so is j'.

(b) Show that if $A_0 \to A_1 \to A_2 \to \ldots$ is a sequence of maps for which every $A_i \to A_{i+1}$ is in $L(F)$, and setting $A := \mathrm{colim}(A_0 \to A_1 \to \ldots)$, show that the map $A_0 \to A$ is in $L(F)$. (More generally, if you know what transfinite composition is, show that $L(F)$ is closed under transfinite compositions.)

(c) Show that $L(F)$ is closed under retracts. That is, given a commutative diagram

$$\begin{tikzcd} A_0' \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & A_0' \arrow[d] \\ A' \arrow[r] & A \arrow[r] & A' \end{tikzcd}$$

where the horizontal composition equals the identity maps, if j is in $L(F)$ then j' is in $L(F)$.

IV.5 Weakly saturated collections

Definition IV.5.0.1. We say that a collection of morphisms S is *weakly saturated* if S is closed under pushouts, transfinite compositions, and retracts.

(a) Show that if S and T are weakly saturated, then so is $S \cap T$. In particular, given a collection A of morphisms, show there exists a *smallest* weakly saturated collection of morphisms containing A. We call this the weakly saturated class of morphisms *generated* by A.

(b) Consider the collection of all boundary inclusions $\partial\Delta^n \subset \Delta^n$ of simplices. Show that the weakly saturated class of morphisms generated by these boundary inclusions is the collection of all injections of simplicial sets.

(c) Show that any inclusion of simplicial sets satisfies the right lifting property for any trivial fibration.

(d) Let F be the class of inner fibrations. Show that the smallest weakly saturated collection containing all inner horn inclusions is a subset of $L(F)$.

Remark IV.5.0.2. Quillen's small object argument shows that the opposite inclusion is true – $L(F)$ *is* the smallest weakly saturated set containing all inner horn inclusions.

Remark IV.5.0.3. You may wonder about the word "weak." Saturation (as opposed to weak saturation) refers to the closure of S under arbitrary coproducts. Standard axioms of set theory render any arbitrary coproduct of morphisms into a transfinite composition (by well-ordering the set indexing the coproduct). Regarding this parenthetical point: It is a consequence of the ZFC axioms that such a well-ordering exists, and it is consistent with ZFC that a definable well-ordering exists – but ZFC is not enough to prove the existence of a definable well-ordering.

IV.6 Slice categories more generally

This exercise is a (very important) generalization of Exercise IV.3.

Let K be a simplicial set and $\mathcal{C}$ an ∞-category. Fix a functor $f : K \to \mathcal{C}$. We define the simplicial set

$$\mathcal{C}_{/f}$$

to have k-simplices given by maps $\Delta^k \star K \to \mathcal{C}$ whose restriction to K agrees with f.

(a) Show that for any $m, n \geq 0$, and $0 < i \leq m$, the inclusion

$$\Lambda^m_i \star \Delta^n \bigcup_{\Lambda^m_i \star \partial\Delta^n} \Delta^m \star \partial\Delta^n \hookrightarrow \Delta^m \star \Delta^n$$

is an inner anodyne map. (This is not a formal fact, and the proof comes down to careful combinatorics. Hint: The inclusion in question can be identified with a much more familiar inclusion.)

(b) Show that for any $m \geq 0$, $0 < i \leq m$, and any inclusion $B_0 \hookrightarrow B$ of simplicial sets, the inclusion

$$\Lambda^m_i \star B \bigcup_{\Lambda^m_i \star B_0} \Delta^m \star B_0 \hookrightarrow \Delta^m \star B$$

is an inner anodyne map. (Hint: Show that the collection of such inclusions $B_0 \hookrightarrow B$ is weakly saturated. This requires you to be a little careful with computing pushouts of pushouts.)

(c) Show that for any $f : K \to \mathcal{C}$, the forgetful functor $\mathcal{C}_{/f} \to \mathcal{C}$ – sending a k-simplex $\Delta^k \star K \to \mathcal{C}$ to its restriction $\Delta^k \to \mathcal{C}$ – is a right Kan fibration.

(d) Show that $\mathcal{C}_{/f}$ is an ∞-category.

IV.7 Compositions are unique up to homotopy (in fact, up to contractible choice of homotopy)

To be a "category" one should require some notion of composition. We've seen than the ability to fill the horn Λ^2_1 gives one intuition that ∞-categories have some composition – given two arrows, we can exhibit a third.

The following (due to Joyal) establishes that this composition is well-defined up to contractible choice.

Theorem IV.7.0.1. Let S be an simplicial set. Then S is an ∞-category if and only if the map

$$\mathsf{Fun}(\Delta^2, S) \to \mathsf{Fun}(\Lambda^2, S)$$

is a trivial fibration.

(a) (Difficult.) Show that the collection of inner anodyne maps is the weakly saturated collection of morphisms generated by the inclusions

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\Delta^m \times \Delta^2) \hookrightarrow \Delta^m \times \Delta^2.$$

(b) Prove Theorem IV.7.0.1.

Lecture V

Thickened manifolds and spaces over BO

We began week two with a result that seems well-known to experts: The homotopy theory of thickened, compact manifolds (up to isotopy equivalence) is equivalent to the homotopy theory of finite CW complexes over BO. While there are proofs involving h-principles (e.g., the Smale-Hirsch theorem) we explained a proof that instead shows that thickened manifolds admit pushouts in the infinity-categorical sense. (They do not admit pushouts in any reasonable, classical sense.) This was inspired by ideas of Oleg Lazarev.

This was also an excuse to introduce, as a baby step, the idea of colimits in an infinity-category (and their utility). Indeed, the proof sketched here uses no Smale-Hirsch theorem – though it does use smooth approximation of continuous functions.

We remark that there is a highly subtle point that we did not touch upon (and did not need to touch upon) in this work. The theory of isotopy equivalences of manifolds with corners and boundaries is very different from the theory of diffeomorphisms of manifolds with corners and boundaries. This is especially true for compact manifolds. Indeed, if one is concerned about diffeomorphisms of compact manifolds, the analogue of Mazur's theorem becomes much more interesting, opening the door to Waldhausen's famous results relating spaces of manifolds to algebraic K-theory.

We also defined in this lecture the notion of equivalence of ∞-categories. This was done a bit quickly. We intuited that for any ∞-category $\mathcal{C}$, and for any object $c \in \mathcal{C}$, the right fibration $\mathcal{C}_{/c} \to \mathcal{C}$ is some model for the Yoneda embedding of c. Since a functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of right fibrations

$\mathcal{C}_{/c} \to \mathcal{D}_{/F(d)}$ over F, one has induced maps of fibers over c' and $F(c')$. If, regardless of c and c', this induced map is an equivalence of Kan complexes[1] we say that F is fully faithful.) Note this definition could also have been given using the left fibrations $\mathcal{C}_{c/}$ or using other models of mapping spaces; any two reasonable definition of full faithfulness are equivalent.

Chapter 7 of Kerodon serves as a readable reference for colimits in infinity-categories.

The result presented here is contained in [17]. For the relation to Waldhausen's work, we recommend his outline [22].

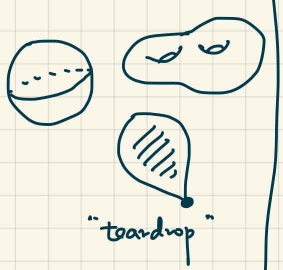
[1] These Kan complexes morally represent the mapping spaces $\hom_{\mathcal{C}}(c', c)$ and $\hom_{\mathcal{D}}(F(c'), F(c))$.

2023-8-7 (月) HA in G ⑤

Spaces over BO are thickened mfd.s

(arXiv: 2307.09647)

Consider Mfld_d



- Obj.s are cpt. C^∞ mfd.s of dim $= d$ (possibly w/ ∂ and $\angle$).
  of any codim.
- $\hom_{\mathrm{Mfld}_d}(X, Y)$ $:= \{ C^\infty \text{ emb.s } X \xrightarrow{j} Y \}$
  ※ j need not respect stratifications in any way.

↑ a top. sp., or a Kan cpx.

We have
$$\begin{aligned} \mathrm{Mfld}_d &\longrightarrow \mathrm{Mfld}_{d+1} . \\ X &\longmapsto X \times [0,1] \\ j &\longmapsto j \times \mathrm{id}_{[0,1]} \end{aligned}$$

<u>Def.</u> $\mathrm{Mfld}^{\diamond} := \bigcup_{d \geq 0} \mathrm{Mfld}_d .$

<u>Rmk.</u> $\mathrm{Mfld}^{\diamond}$ is the category of "stabilized mfd.s" (thickened mfd.s), where $X \sim X \times [0,1]$,

and $\hom_{\mathrm{Mfld}^{\diamond}}(X, Y)$ is

the space of codim-zero emb.s

(up to stabilization).

<u>Observation</u>:

the classifying space of rank d vect. bdles.

$$
\begin{tikzcd}
X \arrow[r, "TX"] & BO(d) \arrow[d, "\oplus \underline{\mathbb{R}}"] \\
X \times [0,1] \arrow[u] \arrow[r, "T(X\times[0,1])"'] & BO(d+1)
\end{tikzcd}
\qquad BO(d) \simeq \{ V \subset \mathbb{R}^{\infty} : \text{subsp.s of } \dim = d \}
$$

up to htpy.

$\rightsquigarrow$ Every object in $\mathrm{Mfld}^{\diamond}$ has a map to $BO = BO(\infty) = \bigcup_{d \geq 0} BO(d)$.

i.e. we have a functor of ∞-cat.s

$$N(\mathrm{Mfld}^{\diamond}) \xrightarrow{(\star)} \mathcal{T}op^{\text{finite}}_{/BO}$$

$$X \longmapsto (X \xrightarrow{TX} BO)$$

<u>Another model</u>:

Given X, consider

$$O = O(\infty) \hookrightarrow \mathrm{Fr}^{\diamond}(X)$$
$$\downarrow$$
$$X$$

← the stabilized frame bundle / X.

(This is principal.)

Also, given $j : X \to Y$ codim $= 0$,

the induced map

$$\mathrm{Fr}^{\diamond}(X) \xrightarrow{Dj} \mathrm{Fr}^{\diamond}(Y)$$

is O-equiv.

$\rightsquigarrow$

$$\begin{tikzcd}
\mathrm{Fr}^{\diamond}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Fr}^{\diamond}(Y) \arrow[d] \\
O \curvearrowright \underset{(\simeq EO)}{\mathrm{pt.}} \arrow[r, equal] & \underset{(\simeq EO)}{\mathrm{pt.}} \curvearrowleft O
\end{tikzcd}$$

: O-equiv.

htpy quot. $/O$ $\rightsquigarrow$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "TX"'] & Y \arrow[d, "TY"] \\
BO \arrow[r, equal] & BO
\end{tikzcd}$$

Thm. ($\star$) is an equivalence of ∞-cats.

(Rmk. "well-known" to experts.)

"Equivalence"

Recall

∞-cat.s

Def. A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is an equivalence if

(1) F is essencially surjective

(i.e. $\forall d \in ob(\mathcal{D})$, $\exists c \in ob(\mathcal{D})$ s.t. $F(c) \cong d$.) ;

(2) F is fully faithful

(i.e. $\forall c_0, c_1 \in ob(\mathcal{C})$, $hom_{\mathcal{C}}(c_0, c_1) \xrightarrow{F} hom_{\mathcal{D}}(F(c_0), F(c_1))$ is a bijection).

i.e. ...

($\exists \alpha, \beta \in \mathcal{D}_1$ and the following triangles :

$$\begin{tikzcd} & F(c) \\ F(c) \arrow[ur, equal] \arrow[r, "\alpha"'] & d \arrow[u, "\beta"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \qquad \begin{tikzcd} & d \\ d \arrow[ur, equal] \arrow[r, "\beta"'] & F(c) \arrow[u, "\alpha"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

)

replace:

$$\left(\forall c \in \mathcal{C}_0, \quad \left(\begin{tikzcd} \mathcal{C}_{/c} \arrow[r, "\text{``}F\text{''}"] \arrow[d] & \mathcal{D}_{/F(c)} \arrow[d] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd} \right) \text{ induces htpy equiv. (of hom spaces).} \right)$$

Eg. Fix X, Y cpt. C^∞ mfd.s.

If $\exists$ a htpy invertible map

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr] & Y \arrow[d] \\ & BO \end{tikzcd} \text{, up to htpy.}$$

C^0

(i.e., $\exists h : Y \to X$ s.t. we have

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr] & Y \arrow[r, "h"] \arrow[d] & X \arrow[dl] \\ & BO & \end{tikzcd} \quad \& \quad \begin{tikzcd} Y \arrow[r, "h"] \arrow[dr] & X \arrow[r, "g"] \arrow[d] & Y \arrow[dl] \\ & BO & \end{tikzcd}$$

with $X = X$ and $Y = Y$ in Top.)

then $\exists$ C^∞ emb.s

$$X \times [0,1]^M \underset{\tilde{h}}{\overset{\tilde{g}}{\rightleftarrows}} Y \times [0,1]^N \quad (\text{codim} = 0)$$

s.t. $\begin{cases} \tilde{g} \circ \tilde{h} \underset{\text{isot.}}{\simeq} id_{Y \times [0,1]^N} \\ \tilde{h} \circ \tilde{g} \underset{\text{isot.}}{\simeq} id_{X \times [0,1]^M}. \end{cases}$

//

Cor: (cf. B. Mazur, Waldhausen " An outline of ... ")

$\boxed{\begin{array}{l}\text{The space of stabilized mfd.s} \\ \cong \text{ the space of spaces } / BO \\ \qquad\qquad \left(\begin{array}{l}\text{htpy. equiv.} \\ \text{to finite cwcpx.s}\end{array}\right)\end{array}}$

⚠ Pending the distinction between

$$X \times \mathbb{R}^M \xrightarrow{\text{Diffeo.}} Y \times \mathbb{R}^N \quad (\text{Mazur, Wald.})$$

v.s.

$$X \times [0,1]^M \xrightarrow[\text{isot. eq.}]{} Y \times [0,1]^N .$$

Construction :

Given an ∞-cat. $\mathcal{C}$,

we can " throwout " all non-isom.s to create

$$\mathcal{C}^{\simeq} \subset \mathcal{C},$$

where $(\mathcal{C}^{\simeq})_0 = \mathcal{C}_0$,

$(\mathcal{C}^{\simeq})_1 = \{ f : c_0 \xrightarrow{\simeq} c_1 \}$,

$(\mathcal{C}^{\simeq})_k = \{ k\text{-spx.s where edegs are all in } (\mathcal{C}^{\simeq})_1 \}$.

Exc. $\mathcal{C}^{\simeq}$ is a Kan cpx.

Def. $\mathcal{C}^{\simeq}$ is the space of objects of $\mathcal{C}$.

E.g. If $\mathcal{C} = \mathcal{T}op$., then

$$\mathcal{C}^{\simeq} \underset{\text{htpy. eq.}}{\overset{!}{\simeq}} \coprod_{\text{htpy eq. classes } [X]} \underbrace{B\,hAut(X)}_{B \text{ of htpy. autom.s of } X}.$$

Also,

$$N(Mfld^{\diamond})^{\simeq} \overset{!}{\simeq} \coprod_{\substack{\text{isot. eq. classes} \\ \lim_{M\to\infty} [X \times [0,1]^M]}} B\,IsotAut(\lim_{M\to\infty} X \times [0,1]^M).$$

$\wr|$ Thm.

$$\left(\mathcal{T}op^{\text{finite}}{}_{/BO}\right)^{\simeq}$$

How to prove Thm.?

(I) Every (stabilized) mfd. is "generated" by $pt = \mathbb{R}^0$.

Handlebody Construction!

- pt. $\sim [0,1] \sim [0,1]^2 \sim \cdots$
- $$\begin{tikzcd} {[0,1]^N} \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & D^N \arrow[r, hook] & {[0,1]^N} \end{tikzcd}$$ ($\wr|$ isot.)

Comment

$\rightsquigarrow$ pt. $\sim D^N$ ($\forall N$),

So we can construct handlebodies!

(II) The same holds for $\mathrm{Top}^{\text{finite}}/BO$. CW cnstxn!

(III) Argue that (★) preserves the "generating operations" (e.g. Handles ⟷ Cells). are all pushouts! ⇝ We can compute hom spaces!

Use smooth approx.

⇝ (★) is ess. surj. !

(IV) Prove $\forall Y$,

$$\hom_{\mathrm{Mfld}^\circ}(pt, Y) \xrightarrow[\simeq]{\text{★}} \hom_{\mathrm{Top}/BO}(pt, Y). \quad \blacksquare$$

Proof of IV $d := \dim Y$.

$$\begin{array}{ccc} j \in \hom_{\mathrm{Mfld}_d}([0,1]^d, Y) & \longrightarrow & \hom_{\mathrm{Mfld}^\circ}(pt, Y) \\ \downarrow \mathrm{ev}: \text{htpy equiv.!} & & \cong \\ \{(y_0 \in Y,\ \mathbb{R}^d \xrightarrow[\cong]{} T_{y_0}Y)\} & & \mathrm{Fr}^\diamond(Y) \end{array}$$

$j \mapsto (j(0), dj_0)$

OTOH, $\hom_{\mathrm{Top}/BO}(pt, Y \xrightarrow{TY} BO)$

$$\simeq \left\{ \begin{tikzcd} & Y \arrow[d, "TY"] \\ pt \arrow[ur, "y_0"] \arrow[r] \arrow[ur, phantom, "\wr"] & BO \end{tikzcd} \right\}$$

$$\simeq \{(y_0 \in Y,\ \text{triv. of } T_{y_0}Y)\}.$$

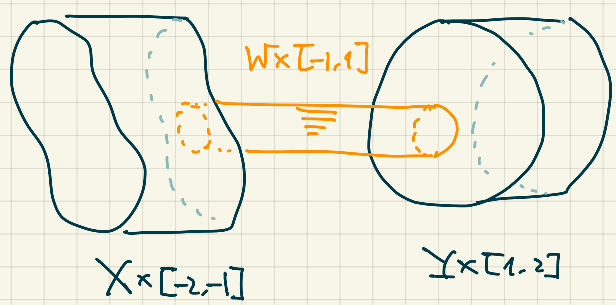
//

Proof of III

Main lemma (O. Lazarev)

Fix C^∞ embs $X \xleftarrow{i} W \xrightarrow{j} Y$ (codim $=0$).

Consider

$P =$ 

and the diagram

$$\begin{tikzcd}
W\times[0,1] \arrow[r, hook, "i\times id"] \arrow[d, hook, "j\times id"'] \arrow[dr, phantom, "\text{up to isot.}"] & X\times[0,1] \arrow[d] \\
Y\times[0,1] \arrow[r, hook] & P
\end{tikzcd}$$

in the ∞-cat $\mathrm{Mfld}^{\diamond}$.

This is a colimit diagram!
(pushout)

※ Let $\mathcal{C}$ be an ∞-cat., and

$\Lambda^2_0 \xrightarrow{f} \mathcal{C}$ be a functor (of ∞-cat.s).

The pushout (colimit) of f is an object $X \in (\mathcal{C}_{f/})_0$

$$f:\ \begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

s.t.

$$\begin{tikzcd} (\mathcal{C}_{f/})_{X/} \arrow[d] \\ \mathcal{C}_{f/} \end{tikzcd}$$

is a trivial fibration

$$\left(\text{i.e. } \forall n \geq 0, \quad (\star):\ \begin{tikzcd} \partial\Delta^n \arrow[r, "\forall"] \arrow[d, hook] & (\mathcal{C}_{f/})_{X/} \arrow[d] \\ \Delta^n \arrow[r, "\forall"'] \arrow[ur, dotted, "\exists"'] & \mathcal{C}_{f/} \end{tikzcd} \right).$$

Recall:

$$\left\{ (\mathcal{C}_{f/})_k = \left\{ \Lambda^2_0 \star \Delta^k \xrightarrow{\bar{f}} \mathcal{C} \;\middle|\; \bar{f}|_{\Lambda^2_0} = f \right\}. \right\}$$

$\underline{n=0}$: $(\star) \Leftrightarrow \forall$ diag.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr] & X_1 \arrow[d] \\ X_2 \arrow[r] & Y \end{tikzcd} \quad f$$

$\exists$ a diag.

$$\begin{tikzcd} & & & X \arrow[dd, dashed] \\ X_1 \arrow[urrr] \arrow[drrr] & X_0 \arrow[l] \arrow[r] \arrow[urr] \arrow[drr] & X_2 \arrow[ur] \arrow[dr] & \\ & & & Y \end{tikzcd}$$

f

"ordinary universality"!

from $\Lambda^2_0 \star \Delta^1$.

We call ($\star$) for $^\forall n \geq 1$
"uniqueness up to contractible choice".

Rmk. (Classical proof)

Even without ∞-cats, we already had ($\star$).

Ess. surj.? :

Given $\underset{\text{CW finite}}{X} \xrightarrow{E} BO$.

embed $X \subset \mathbb{R}^N$;

take a nbd. $X \subset \mathrm{Nbd}(X) \underset{\text{open}}{\subset} \mathbb{R}^N$ w/ C^∞ body,

so $\overline{\mathrm{Nbd}(X)}$ is a cpt. C^∞ mfd..

$\rightsquigarrow \mathrm{Disk}(\overline{E}) \subset \overline{E}$

Fully faithful? : Smale–Hirsch theory!
(h-principle) $\blacksquare$

Exercises: Colimits

V.1 Definition of colimit

Fix a simplicial set K and a functor $f : K \to \mathcal{C}$. We defined the right fibration $\mathcal{C}_{/f}$ in Exercise IV.6.

Dually, we can define the *undercategory* (really, under-∞-category)

$$\mathcal{C}_{f/}$$

by setting its k-simplices to consist of maps $K \star \Delta^k \to \mathcal{C}$ that restrict to f along K. Note that an object of this ∞-category is some object of $\mathcal{C}$ receiving a map "from f."

(If you like, you can take $K = \Lambda^2_0$ in this example. To visualize things, it may help to note that $K \star \Delta^0 \cong \Delta^1 \star \Delta^1$.)

Definition V.1.0.1 (Colimit)**.** Fix an object $x \in \mathcal{C}_{f/}$. We say that x is a *colimit of* f if the fibration

$$(\mathcal{C}_{f/})_{x/} \to \mathcal{C}_{f/}$$

is a trivial fibration.

When $K = \Lambda^2_0$, a colimit for f is called a *pushout* of f.

(a) By using the lifting property for the inclusion $\emptyset = \partial\Delta^0 \to \Delta^0$, show that for any object $y \in \mathcal{C}_{f/}$, you can find a map from y to x in $\mathcal{C}_{f/}$.

(b) By using the lifting property for the inclusion $\partial\Delta^1 \cong \Delta^0 \coprod \Delta^0 \to \Delta^1$, show that any two maps as in the previous exercise are homotopic.

(c) Show that any two homotopies as in the previous exercise are in fact related by a higher homotopy (e.g., by exhibiting an appropriate 2-simplex in $\mathcal{C}_{f/}$).

(d) Suppose that both x and x' are colimits of f. Show that there is an isomorphism $x \cong x'$ in $\mathcal{C}_{f/}$.

V.2 An example: Pushout in sets

This is a warm-up using sets (either for people not used to pushouts, or for people who want to translate classically familiar notions to ∞-categorical language).

Let $\mathcal{C} = \mathsf{Sets}$ be the (nerve of the) category of sets, and fix a diagram $f : \Lambda^2_0 \to \mathcal{C}$. Concretely, this is the data of three sets W, X, Y and functions

$$X \leftarrow W \to Y.$$

(a) Consider the set $X \bigcup_W Y$, defined as the quotient set

$$X \bigcup_W Y := (X \coprod Y)/ \sim$$

where we say $x \sim y$ if there exists an element w whose images in X and Y are equal to x and y, respectively. Show that the functions

$$X \to X \bigcup_W Y, \qquad x \mapsto [x], \qquad Y \to X \bigcup_W Y, \qquad y \mapsto [y]$$

defines an object of $\mathcal{C}_{f/}$.

(b) Show that $X \bigcup_W Y$ (considered as an object of $\mathcal{C}_{f/}$ as in the previous exercise) is a colimit for f.

(c) For any set Z, exhibit a bijection

$$\hom(X \bigcup_W Y, Z) \to \hom(X, Z) \times_{\hom(W,Z)} \hom(Y, Z)$$

where $\hom = \hom_{\mathsf{Sets}}$ denotes the set of functions.

V.3 An example: Mapping cones in cochain complexes

This is a more ∞-categorical example.

Fix a base ring R and fix a chain map $f : A \to B$ between two R-linear cochain complexes. We define the *mapping cone* of f to be the cochain complex

$$\mathrm{Cone}(f) := B \oplus A[1], \qquad d(b,a) := (db + (-1)^{|a|} f(a), da).$$

(a) Verify that $\mathrm{Cone}(f)$ is a cochain complex.

(b) Consider the diagram $\Lambda^2_0 \to \mathsf{Chain}_R$ to the dg-nerve of the dg-category of R-linear chain complexes given by

$$0 \leftarrow A \xrightarrow{f} B.$$

Exhibit a functor $\Lambda^2_0 \star \Delta^0 \to \mathsf{Chain}_R$ sending the vertex Δ^0 to $\mathrm{Cone}(f)$. By abuse of notation, we will also notate this functor by $\mathrm{Cone}(f)$. It is an object of $(\mathsf{Chain}_R)_{f/}$.

(c) Show that the functor $((\mathsf{Chain}_R)_{f/})_{\mathrm{Cone}(f)/} \to (\mathsf{Chain}_R)_{f/}$ has the right lifting property with respect to all inclusions $\emptyset = \partial\Delta^0 \to \Delta^0$.

(d) Show that the functor $((\mathsf{Chain}_R)_{f/})_{\mathrm{Cone}(f)/} \to (\mathsf{Chain}_R)_{f/}$ has the right lifting property with respect to all inclusions $\partial\Delta^1 \to \Delta^1$.

(e) Show that the functor $((\mathsf{Chain}_R)_{f/})_{\mathrm{Cone}(f)/} \to (\mathsf{Chain}_R)_{f/}$ has the right lifting property with respect to all inclusions $\partial\Delta^2 \to \Delta^2$.

(f) (*) Show that $\mathrm{Cone}(f)$ is a pushout.

Lecture VI

Stabilized Weinstein sectors

The second lecture of week two introduced the idea of localizations. The main takeaway is that even classical categories admit infinity-categorical localizations that encode rich information. Here is the flagship example : A localization of the (ordinary) category of reasonable topological spaces recovers the infinity-category of topological spaces.

We then tried to explain that a similar result holds for a class of nicely-behaved symplectic manifolds: Weinstein sectors. (This is joint work with Oleg Lazarev and Zach Sylvan.) The references are [6, 7]. For those interested in wrapped Fukaya categories, we find that the most user-friendly definition is contained in [4]. Finally, Section 6.3 of Kerodon contains material on localizations of infinity-categories.

2023-8-8 (火) HA in G ⑥

The ∞-category of stabilized Weinstein sectors

(jt. w/ O. Lazarev, Z. Sylvan)

arXiv: 2110.11754, 2109.06069.

Last time : utility of colimits

Today : utility of localizations

Recall : (Algebraic Geometry)

$$\begin{array}{ccc} \mathbb{C}[t] & \longleftrightarrow & \mathbb{A}^1_{\mathbb{C}} . \\ \text{Localization} \downarrow & & \downarrow \text{Localization} \\ \mathbb{C}[t, t^{-1}] & \longleftrightarrow & \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} . \end{array}$$

This Motivates why " inverting something multiplicatively " is called localization.

In a category, composition should be thought of as a multiplicative operation.

Algebraically localizations satisfy a "universal property":

Fix a comm. ring R and a cllxn $S \subset R$.

$$\begin{tikzcd} R \arrow[r, "\forall f"] \arrow[d] & \forall R' \\ R[S^{-1}] \arrow[ur, dotted, "\exists !"'] & \end{tikzcd} \quad \text{s.t. } f(S) \subset (R')^{\times}$$

$R[S^{-1}]$: the localization of R along S.

In the theory of cat.s ... :

$$\left(\coprod_{x,y \in \mathrm{ob}\,\mathcal{C}} \hom_{\mathcal{C}}(x,y) \right) \; = \; \hom_{\mathcal{C}}$$

Fix a cat. $\mathcal{C}$ and a cllxn. $S \subset \hom_{\mathcal{C}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\forall f"] \arrow[d] & \forall \mathcal{C}' \\ \mathcal{C}[S^{-1}] \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \quad \text{s.t. } \forall s \in S,\ f(s) \text{ is isom.}$$

$\exists$ (unique up to natural isom.)

In the theory of ∞-cat.s ... :

Fix an ∞-cat. $\mathcal{C}$ and a cllxn $S \subset \mathcal{C}_1$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\forall f"] \arrow[d] \arrow[dr, phantom, "\sim"] & \forall \mathcal{C}' \\ \mathcal{C}[S^{-1}] \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \quad \text{s.t. } \forall s \in S,\ f(s) \text{ is isom.}$$

$\exists$ (unique up to contractible choice)

Remarkable fact :

Even when $\mathcal{C}$ is (the nerve of) an ordinary category, the localization $\mathcal{C}[S^{-1}]$ (taken in $\mathcal{C}at_\infty$) can have an interesting topology (in the hom space) !

E.g. / Thm. (Quillen, Kan, ...)

Let Top = ordinary cat. of top. sp.s ($\simeq \exists$ CW cpx.s),

$S :=$ eqs $:= \{$ htpy equiv.s $\}$.

$$\begin{tikzcd} N(\mathrm{Top}) \arrow[r] \arrow[d] & \mathcal{T}op \\ N(\mathrm{Top})[\mathrm{eqs}] \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

$\leadsto$ Its $\hom(X,Y) \simeq C^0(X,Y)$ w/ c.o. top.

$\exists$ (unique up to cont. choice)

Thm. : This is an equiv. !

Cor. Suppose

$W : \mathrm{Top} \to \mathcal{D}$ any ∞-cat., say chain cpx.s$/\Bbbk$.

is a functor

s.t. $\forall f : X \to Y$: h.eq.,

$W(f) : W(X) \to W(Y)$ is an isom.

Then $\forall X \in \mathrm{Top}$, $\mathcal{F}(X)$ recieves

a (htpy coherent) action by $h\mathrm{Aut}(X)$.

Rmk. (cf. cofib., model cat.)

Often, $\mathcal{C}[S^{-1}] \rightsquigarrow ho(\mathcal{C}[S^{-1}])$

same obj.,

hom $:= \pi_0(\hom_{\mathcal{C}[S^{-1}]})$

recovers

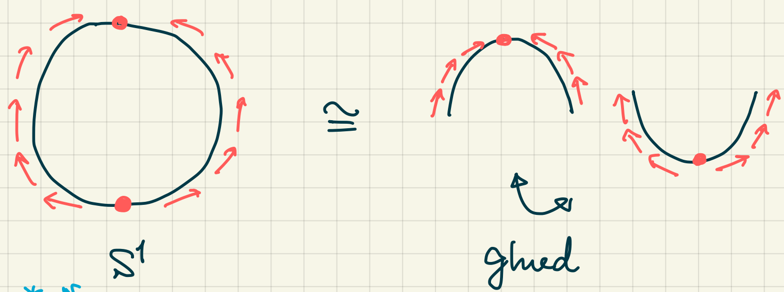
the 1-cat. localiz. we seek.

It is often involved to make family versions of analytic invariants (SW inv.s, Floer, ...).

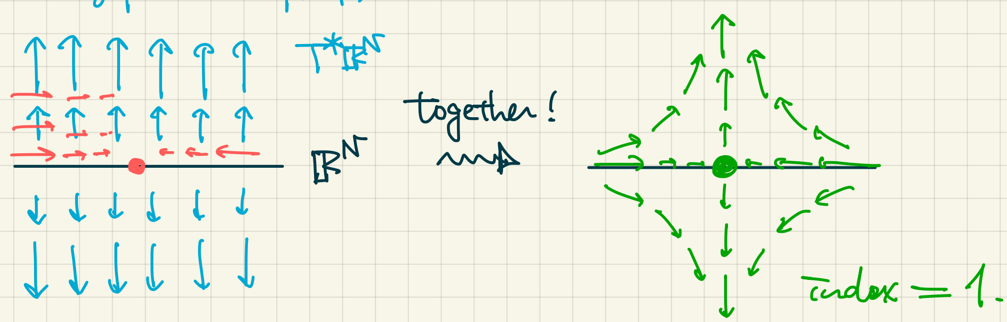
Today, we will exhibit group actions on various sympl. invariants (e.g. wrapped Fukaya categories) on nice sympl. mfd.s !

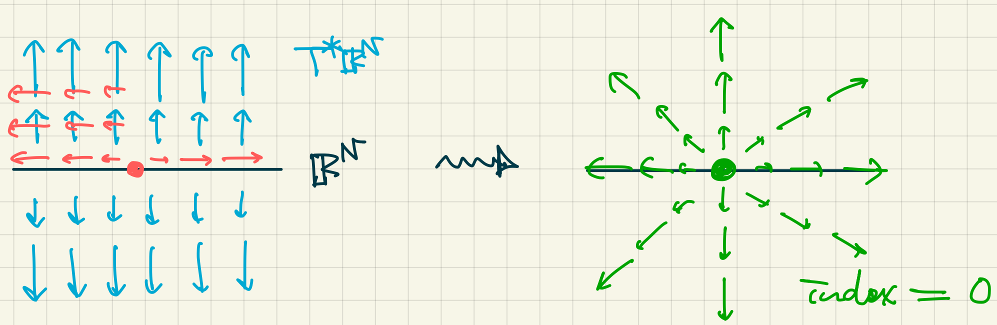
Introduction to Weinstein sectors :

Recall Morse Theory :



$T^*\mathbb{R}^N \ni (q,p) \mapsto (0, |p|^2 \cdot p)$





$T^*\mathbb{R}^N$ is sympl., i.e., equipped with

$\omega \in \Omega^2(T^*\mathbb{R}^N)$ non-degen.

If (X, ω) is sympl., then we have

$$
\begin{array}{ccc}
TX & \xrightarrow{\cong} & T^*X \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
v & \longmapsto & \omega(v,-)
\end{array}
$$

$$\rightsquigarrow$$

$$
\begin{array}{ccccc}
\mathfrak{X}(X) = & \Gamma(TX) & \xrightarrow{\cong} & \Gamma(T^*X) & = \Omega^1(X) . \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\
& v & \longmapsto & \theta &
\end{array}
$$

Fact $\quad d\theta = \omega$.

We can glue $T^*\mathbb{R}^N$'s in a Morse thy-like way respecting v

to obtain sympl. mfd.s

w/ a globally defined v

(hence " " θ) s.t. $d\theta = \omega$.

$\rightsquigarrow$ Any sympl. mfd. made this way

is called a Weinstein mfd.

There is also a notion

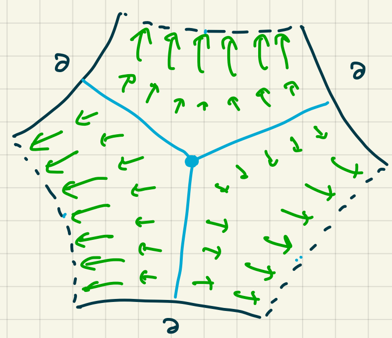
for mfd.s w/ ∂ or $\angle$,

Weinstein sectors.

E.g. Q : a mfd. possibly w/ ∂ or $\angle$.

$\rightsquigarrow$ T^*Q is a Weinstein sector.

(e.g.



)

Rmk. Weinstein sectors are a model for " T^* of singular spaces. "

[See "arborealization".

D. Nadler, L. Starkston,
D. Alvarez-Gavela, Y. Eliashberg.]

Rmk.
$\forall$ Weinstein sector X, we can define

$W(X)$ the wrapped Fukaya category of X.

A_∞-category
(but you can imagine that $W(X)$ is just a dg-cat.)

∞ (For htpy theoritist, A_∞-cat. $\|$ dg-enriched ∞-cat.)

A Weinstein mfd. X has (as part of its data) a 1-form θ.

Define a strict (resp. not necessarily strict) map $j : X \to Y$ to be a C^∞, proper, codim-zero embedding

s.t. $j^*\theta_Y = \theta_X$.

(resp. $j^*\theta_Y = \theta_X + dh$ ($\exists h$: cpt.ly supported))

Def.

$\boxed{\begin{array}{l} \mathrm{Wein}_d^{str} := \text{cat. whose obj are Weinstein sectors of } \dim = d, \\ \hom(X,Y) := \{ j : X \to Y \text{ strict} \}. \end{array}}$

Rmk. (cont'd.) [Ganatra–Pardon–Shende]

$\forall$ (even) d,

(0) $\mathcal{W} : \mathrm{Wein}_d^{str} \to A_\infty\mathrm{Cat}$ is a functor;

(1) $\forall X$, $\mathcal{W}(X) \xrightarrow{\exists} \mathcal{W}(X \times T^*[0,1])$ is an equivalence;

(2) Suppose $j : X \to Y$ is a strict map,

and $\exists h : Y \to X$ a (not nec. strict) map
and
(☆) C^∞ isot.s $h \circ j \sim id$, $j \circ h \sim id$.
through (not nec. strict) maps.

Then $W(j) : W(X) \to W(Y)$ is an equiv.

This motivates :

Def.

① Let $\mathrm{Wein}^{str, \diamondsuit} := \bigcup_{d \geq 0} \mathrm{Wein}^{str}_d$.

$$\begin{aligned} \mathrm{Wein}^{str}_d &\to \mathrm{Wein}^{str}_{d+2} \\ X &\longmapsto X \times T^*[0,1] \\ j &\longmapsto j \times id_{T^*[0,1]} . \end{aligned}$$

※ This is not (easily) defined for j that are not strict.

② Let $\mathrm{eqs} := \{ \text{strict maps } j \text{ satisfying (☆)} \}$.

Thm. (LaST)

$N(\mathrm{Wein}^{str, \diamondsuit})[\mathrm{eqs}^{-1}]$ is equiv. to an ∞-cat.
whose spaces are the spaces of stabilized (not nec. strict) maps.

As a result :

$$
\begin{tikzcd}
\mathrm{Wein}_0^{\mathrm{str}} \arrow[r] \arrow[d, "(0)"'] \arrow[dr, phantom, "(1)"] & \mathrm{Wein}_2^{\mathrm{str}} \arrow[r] \arrow[dl] \arrow[d, phantom, "(1)"] & \mathrm{Wein}_4^{\mathrm{str}} \arrow[r, dotted] \arrow[dll] & \mathrm{Wein}^{\mathrm{str},\diamond} \arrow[dlll, "\exists\ (0)+(1)"] \arrow[d] \\
A_\infty \mathrm{Cat} & & & \mathrm{Wein}^{\mathrm{str},\diamond}[\mathrm{eqs}^{-1}] \arrow[lll, dashed, "\exists\ \left(\text{up to cont. choice}\right)", "(2)"']
\end{tikzcd}
$$

$\left(\text{+ univ. prop. of localizations}\right)$

$A_\infty\mathrm{Cat}$

$\dagger$

∞-cat. of A_∞-cat.s

$\left(\begin{array}{l}\text{localization of} \\ \text{the ord. cat. of } A_\infty\text{-cat.s} \\ \text{w/ cofibrant hom cpx.s}\end{array}\right)$

Cor. $\forall$ Weinstein sector X,

the group of X's

(stabilized, not nec. strict) autom.s

acts coherently on $\mathcal{W}(X)$.

※ A coherent G-action on $\mathcal{W} \in \mathcal{A}^{\infty\text{-cat.}}$

is a functor $BG \to \mathcal{A}$

$* \longmapsto \mathcal{W}$.

(Pf Technique)

Given any ∞-cat. $\mathcal{C}$,

you can make a new ∞-cat. $\mathrm{Ex}_{\simeq}(\mathcal{C})$,

where

$$\mathrm{Ex}_{\simeq}(\mathcal{C})_0 = \mathcal{C}_0 .$$

$$\mathrm{Ex}_{\simeq}(\mathcal{C})_1 = \left\{ X_0 \to X_{01} \xleftarrow{\sim} X_1 \right\}$$

$$\mathrm{Ex}_{\simeq}(\mathcal{C})_2 = \left\{ \begin{tikzcd} & & X_2 \arrow[dl, "\sim"'] \arrow[d, "\sim"] \arrow[dr, "\sim"] & & \\ & X_{02} \arrow[r, "\sim"] & X_{012} & X_{12} \arrow[l, "\sim"'] & \\ X_0 \arrow[ur] \arrow[urr] \arrow[rr] & & X_{01} \arrow[u] & & X_1 \arrow[ll, "\sim"] \arrow[ul] \arrow[ull] \end{tikzcd} \right\}$$

$\vdots$

<u>Thm.</u> $\mathcal{C} \xrightarrow{\sim} \mathrm{Ex}_{\simeq}(\mathcal{C})$.

$$\begin{tikzcd} N(\mathrm{Wein}^{\mathrm{str}.\diamond}) \arrow[r] \arrow[d] & \text{"comb. model"} \arrow[r, Rightarrow, "\sim"] \arrow[d] & \mathrm{Ex}_{\simeq}(\ ) \\ N(\mathrm{Wein}^{\mathrm{str}\diamond})[\mathrm{eqs}^{-1}] \arrow[r, "\sim"] \arrow[ur, dashed, "\exists"'] & \mathrm{Ex}_{\simeq}(\ ) \arrow[ur] & \end{tikzcd}$$

$\blacksquare$

Exercises on localizations

Recall the ∞-category of ∞-categories

$$\mathcal{C}\mathrm{at}_\infty$$

from Exercise III.4.

VI.1 Definition of localization

Let $\mathcal{C}$ be an ∞-category and choose a collection of morphisms $S \subset \mathcal{C}_1$. Let $\mathcal{E}$ be an ∞-category equipped with a functor

$$L : \mathcal{C} \to \mathcal{E}.$$

Definition VI.1.0.1. We say that L *exhibits* $\mathcal{E}$ *as a localization of* $\mathcal{C}$ *along* S, and that L *is a localization of* $\mathcal{C}$ *along* S if for all ∞-categories $\mathcal{D}$, the pullback functor

$$L^* : \mathsf{Fun}(\mathcal{E}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful, and if the essential image of L^* consists of those functors $f : \mathcal{C} \to \mathcal{D}$ sending all morphisms in S to isomorphisms in $\mathcal{D}$.

(a) Suppose $L : \mathcal{C} \to \mathcal{E}$ is a localization of $\mathcal{C}$ along S. Show that for any functor $f : \mathcal{C} \to \mathcal{D}$ for which $f(S) \subset (\mathcal{D}^{\simeq})_1$, there exists a diagram $\Delta^2 \to \mathcal{C}\mathrm{at}_\infty$ in the ∞-category of ∞-categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "L"'] & \mathcal{D} . \\ \mathcal{E} \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

(b) Given two diagrams as above (and in particular, potentially two edges $\tilde{f}$ and $\tilde{f}' : \mathcal{E} \to \mathcal{D}$) show that the dashed edges are homotopic. (That is, show there exists a 2-simplex in $\mathcal{C}\mathrm{at}_\infty$ with edges given by $\tilde{f}, \tilde{f}'$, and a degenerate edge.)

VI.2 An example

(a) (Way too hard.) Suppose $\mathcal{C}$ and $\mathcal{E}$ are (nerves of) ordinary categories, and suppose $L : \mathcal{C} \to \mathcal{E}$ admits a fully faithful right adjoint (in the classical sense). Convince yourself that L is a localization (of ∞-categories).

(b) Fix $n \geq 0$. Let $\mathcal{P}'([n])$ be the partially ordered set of non-empty subsets of $[n]$, ordered by inclusion. Consider the map of posets

$$\mathcal{P}'([n]) \to [n], \qquad I \mapsto \max I.$$

Show that (the nerve of) this map is a localization.

Lecture VII

Factorization homology

The third lecture of week two introduced the audience to the theory of E_n-algebras and the invariants of n-dimensional manifolds (factorization homology) one can make out of these algebras. The idea that a hierarchy of commutativity, from E_1 to E_∞, exists in homotopy theory was by now a recurring theme in our lectures.

We did not delve into the full complexity of the myriad infinity-categorical tools that go into the foundations of factorization homology, but we gave a rough outline of some of the first theorems of the field.

We ended by saying that one we can encode algebras and modules using factorization homology for stratified spaces, it is natural to wonder how to capture the theory of higher categories using stratified spaces. This is one direction that the works of Ayala-Francis, Ayala-Francis Rozenblyum, and Ayala-Mazel-Gee-Rozenblyum pursue. A famous conjecture relating categorical notions with manifold-theoretic notions is the cobordism hypothesis, and indeed a large initial motivation for factorization homology was to give an alternative proof strategy for this conjecture.

We present no exercises. For further reading, we refer the reader to the book [19], a freely available version of which is on the arXiv [18]. For an even more global perspective, we refer to Ayala-Francis's primer [1].

2023-8-9 (GK) HA in G ⑦

Introduction to Factorization Homology

a.k.a. topological chiral homology (Lurie)

jt w/ D. Ayala, J. Francis

(developed thereafter by
Ayala, Francis, Rozenblyum, Mazel-Gee, ...)

An example:

Consider the category $\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$

$$\begin{cases} \mathrm{ob}(\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}) = \{\phi, \mathbb{R}, \mathbb{R}\amalg\mathbb{R}, \mathbb{R}\amalg\mathbb{R}\amalg\mathbb{R}, \ldots\}, \\ \mathrm{hom}(X, Y) = \{\text{ori. pres. } C^\infty \text{ open emb.s } X \to Y\} \\ \qquad\qquad\qquad\qquad\qquad \text{w/ topology.} \end{cases}$$

$\rightsquigarrow$ $N(\mathcal{D}\mathrm{isk}_1^{\mathrm{or}})$ is an interesting ∞-cat.

Rmk. Sym. $\otimes$ strc. of $\amalg$:

$$(X, Y) \longmapsto X \amalg Y.$$

Fix $\mathcal{V} = \mathrm{Chain}_{\Bbbk}$. (resp. $= \mathcal{C}\mathrm{hain}_{\Bbbk}$) ∞-cat.

Consider a functor (resp. $N(\mathrm{Disk}_1^{or})$) ← これも Disk_1^{or} と略す.（文脈から大体分かる）

$$A : \mathrm{Disk}_1^{or} \longrightarrow \mathcal{V}$$

equipped w/ data of isom.s

$$A(X \amalg Y) \xleftarrow{\cong} A(X) \otimes_{\Bbbk} A(Y).$$

[i.e. A is a sym. monoidal functor.]

Further assume

$$j \underset{\text{isot.}}{\sim} j' \Rightarrow A(j) = A(j').$$

[resp. As a conseq., $\hom(X, Y) \to \hom(A(X), A(Y))$ respects htpies (& higher ones)]

e.g. $j \overset{H}{\sim} j' \Rightarrow A(j) \overset{A(H)}{\sim} A(j')$.

Prop. The data of A is equivalent to the data of an ass. $\Bbbk$-alg. (A_∞)

More precisely, $\exists$ an equiv. of (∞-) cat.s

$$\{ \text{Sym} \otimes \text{ functors } A : \mathrm{Disk}_1^{or} \to \mathcal{V} \} \xrightarrow{\sim} \{ \text{Ass. } (A_\infty) \ \Bbbk\text{-alg.s} \}$$

(Sketch of proof) Fix A, and put $A(\mathbb{R}) =: \underline{A}$.

A : sym. $\otimes$ $\rightsquigarrow$ $A(\phi) \cong \Bbbk$,

$A(\mathbb{R} \amalg \mathbb{R}) \cong \underline{A} \otimes_{\Bbbk} \underline{A}$.

We have the following :

- $\emptyset \to \mathbb{R} \overset{A}{\longmapsto} \Bbbk \xrightarrow{u} \underline{A}$.
- $\mathbb{R} \sqcup \mathbb{R} \overset{\forall j}{\hookrightarrow} \mathbb{R} \overset{A}{\longmapsto} \underline{A} \otimes_{\Bbbk} \underline{A} \xrightarrow{A(j) =: m_j} \underline{A}$.

$\left(j \overset{H}{\sim} j' \Rightarrow m_j \overset{A(H)}{\sim} m_{j'} \right)$

①
$$\begin{tikzcd}
\emptyset \sqcup \mathbb{R} \arrow[r] \arrow[dr] & \mathbb{R} \sqcup \mathbb{R} \arrow[d, "j"] \arrow[dl, phantom, "\text{isot.}"] \\
& \mathbb{R}
\end{tikzcd}
\overset{A}{\longmapsto}
\begin{tikzcd}
\Bbbk \otimes \underline{A} \arrow[r, "u \otimes id"] \arrow[dr, "id"'] & \underline{A} \otimes \underline{A} \arrow[d, "m_j"] \arrow[dl, phantom, "\circlearrowright"] \\
& \underline{A}
\end{tikzcd}$$

up to htpy

$\therefore$ u is a unit for m.
up to htpy ($\because$ m_j is unital).
htpy

②
$$\begin{tikzcd}
\mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R} \arrow[r, "id \sqcup j"] \arrow[d, "j \sqcup id"'] \arrow[dr, phantom, "\text{isot.}"] & \mathbb{R} \sqcup \mathbb{R} \arrow[d, "j"] \\
\mathbb{R} \sqcup \mathbb{R} \arrow[r, "j"'] & \mathbb{R}
\end{tikzcd}
\overset{A}{\longmapsto}
\begin{tikzcd}
\underline{A}^{\otimes 3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \underline{A}^{\otimes 2} \arrow[d] \\
\underline{A}^{\otimes 2} \arrow[r] & \underline{A}
\end{tikzcd}$$

up to htpy

$\therefore$ m is associative. $\blacksquare$

Question Given A,

is there a natural extension to Mfld_1^{or} ?

- need not cpt.
- without ∂.

Def.

Let $\mathcal{V}$ be a nice sym. $\otimes$ ∞-cat., and (e.g. $\mathcal{V} = \mathrm{Chain}_{\Bbbk}$ w/ $\otimes_{\Bbbk}$)

$A : \mathrm{Disk}_1^{\mathrm{or}} \to \mathcal{V}$ a sym. monoidal functor.

The factorization homology with coeff. A is the LEFT KAN EXTENSION of A along ι:

$$\begin{tikzcd} \mathrm{Disk}_1^{\mathrm{or}} \arrow[r, "A"] \arrow[d, "\iota"'] & \mathcal{V} \\ \mathrm{Mfld}_1^{\mathrm{or}} \arrow[ur, dashed, "\int A"'] & \end{tikzcd}$$

$\rightsquigarrow$ We say $\int_X A := \int A(X)$ is the fact. homol. over X.

Thm. Fix A an A_∞-alg. (ass. alg.)

$\Rightarrow$ $\exists$ a quasi-isom.

$$\int_{S^1} A \simeq A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} A.$$

Hochschild chain cpx. of A.

Rmk. $\mathrm{hom}(S^1, S^1) = \mathrm{Diff}_+(S^1) \simeq S^1$.

$\rightsquigarrow$ $\mathrm{Diff}_+(S^1)\,(\simeq S^1) \curvearrowright\, ? \int_{S^1} A$.

Rmk. If we consider $\mathrm{Disk}_1^{\cancel{\mathrm{or}}}$,

sym. $\otimes$ $\mathcal{A} : \mathrm{Disk}_1 \longrightarrow \mathcal{V}$

$\xleftrightarrow{\text{equiv.}}$ an. A_∞-alg. $\underline{A}$

w/ an involution $\sigma : \underline{A} \xrightarrow{\sim} \underline{A}$.

<u>In higher dim.s</u> :

Consider functors

$$\mathcal{A} : \mathrm{Disk}_n \longrightarrow \mathcal{V}.$$

obj: ϕ, $\mathbb{R}^n$, $\mathbb{R}^n \sqcup \mathbb{R}^n$, ...

hom: $\mathcal{C}^\infty$ open emb.s

Given $\mathcal{A}$, we define fact. homol. w/ coeff. $\mathcal{A}$ as the left Kan ext.

$$\begin{tikzcd} \mathrm{Disk}_n \arrow[r, "\mathcal{A}"] \arrow[d] & \mathcal{V} \\ \mathrm{Mfld}_n \arrow[ur, dashed, "\int \mathcal{A}"'] & \end{tikzcd}.$$

Q What data goes into $\mathcal{A}$?

$$\left\{ \begin{array}{l} \bullet\ \mathcal{A}(\mathbb{R}^n) =: \underline{A}. \\ \bullet\ \mathcal{A}(\phi \to \mathbb{R}^n) = \Bbbk \xrightarrow{u} \underline{A}. \end{array} \right.$$

; also ...

$$\mathcal{A}\left(\overset{\mathbb{R}^n \sqcup \mathbb{R}^n}{\boxed{1}\ \ \boxed{2}} \xrightarrow{j} \overset{\mathbb{R}^n}{\boxed{\boxed{1}\ \ \boxed{2}}} \right) = \underline{A} \otimes \underline{A} \xrightarrow{m_j} \underline{A}.$$

$$\begin{array}{ccccc} \text{swap}\downarrow & \text{(Isot.)} & \| & \text{swap}\downarrow \ \text{(htpy)} & \| \\ A\Big(\ \boxed{2}\ \ \boxed{1} & \xrightarrow{j} & \boxed{\boxed{2}\ \boxed{1}}\ \Big) & = \ \underline{A}\otimes\underline{A} & \xrightarrow{m_j} \underline{A}\,. \end{array}$$

Informally, a sym. $\otimes$ functor $\mathrm{Disk}_n \to \mathcal{V}$

is the data of an alg. $\underline{A}$

$\mathbb{E}_n$-alg. w/ mult. that commutes up to htpically ambiguity of S^{n-1}.

w/ compatible $O(n)$-action.

framed $\mathbb{E}_n$-alg.

※ $\mathbb{E}_1 = A_\infty$.

<u>Rmk.</u> Define Disk_n^{fr} to be :

obj : ϕ, $\mathbb{R}^n$, $\mathbb{R}^n \amalg \mathbb{R}^n$, ...

equipped w/ the std. framings of $T\mathbb{R}^n \cong \mathbb{R}^n$.

$$\hom(X, Y) = \left\{ \begin{array}{l} C^\infty \text{ open emb.} \\ j : X \to Y \end{array} \text{ w/ data htpy. btwn framings} \right\}.$$

<u>e.g.</u> $\hom_{\mathrm{Disk}_n^{fr}}(\mathbb{R}^n, \mathbb{R}^n) = *$.

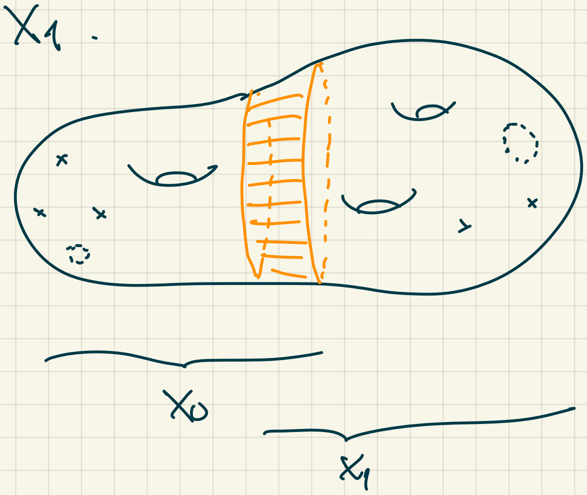
$\left(\text{cf. } \hom_{\mathrm{Disk}_n}(\mathbb{R}^n, \mathbb{R}^n) \simeq O(n). \right)$

<u>Rmk.</u> We can define fact. homol. for Mfld_n^{fr}.

$$\begin{tikzcd}
\mathrm{Mfld}_n^{fr} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Top}_{/(* \to BO(n))} \arrow[d] & \ni \left(\begin{tikzcd}[ampersand replacement=\&] \& * \arrow[d] \\ X \arrow[ur] \arrow[r] \& BO(n) \end{tikzcd} \right) \arrow[d, mapsto] \\
\mathrm{Mfld}_n \arrow[r] & \mathrm{Top}_{/BO(n)} & \\
X \arrow[r, mapsto] & (X \xrightarrow{TX} BO(n)) &
\end{tikzcd}$$

Fact. hom. is, in theory, "computable".

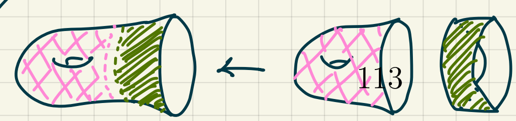
Fix $X = X_0 \underset{W\times\mathbb{R}}{\cup} X_1$.



<u>Obs.</u>

- $\int_{W\times\mathbb{R}} A$ is an $\overset{(\mathbb{E}_1)}{A_\infty}$-alg. $\circlearrowleft$ $\left(\begin{tikzcd} \mathrm{Disk}_1^{or} \arrow[r] \arrow[d] & V \\ \mathrm{Mfld}_n \arrow[ur, bend right] & \end{tikzcd} \quad \mathbb{R} \mapsto W\times\mathbb{R}\right)$
- $\int_{X_i} A$ is a [ ] modules.

right if $i=0$
left if $i=1$.



$$X_0 \longleftarrow X_0 \amalg W\times\mathbb{R}$$

$$\rightsquigarrow$$

$$\int_{X_0} A \longleftarrow \int_{X_0} A \otimes \int_{W\times\mathbb{R}} A.$$

Thm. [AF, AFT] ($\otimes$-excision for fact. hom.)

$\exists$ an equiv.

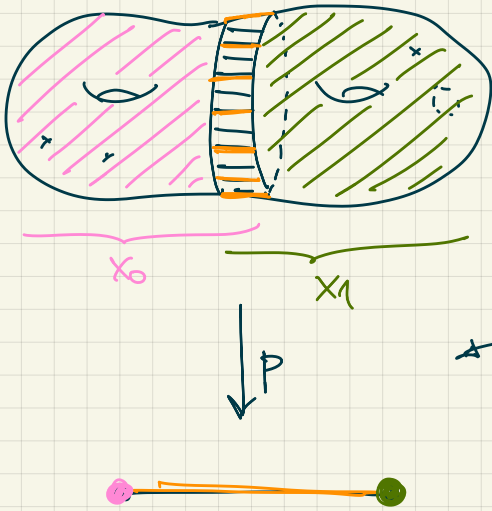
$$\int_X A \simeq \int_{X_0} A \overset{\mathbb{L}}{\underset{\int_{W\times\mathbb{R}} A}{\otimes}} \int_{X_1} A \quad \text{in } \mathcal{V}.$$

Thm. [AF, AFT]

Every sym. $\otimes$ functor $\mathrm{Mfld}_n^{\sqcup} \to \mathcal{V}$ satisfying $\otimes$-excision arises as fact. homol.

(Sketch of proof of $\otimes$-excision)

Pass to fact. hom. for ~~stratified mfd.~~ $[0,1]$!



← p is a "constructible bdle".

<u>Lemma</u> ① Hom. pushes forward along such p.

(<u>e.g.</u> Given Disk $D \xrightarrow{p} [0,1]$,

have

$$p^{-1}(\mathrm{Image}(D)) = \text{some mfd.}$$

$$\Downarrow$$

$$\int_{\substack{\text{same}\\ \text{mfd.}}} A =: \int_D p_* A. \quad)$$

<u>Lemma</u> ②

Data of $\mathrm{Disk}_1^{\bullet\!-,\;-\!\bullet} \longrightarrow \mathcal{V}$ determines M_0, M_1 over A.

$\downarrow$

$\mathrm{Mfld}_1 \ni \bullet\!\!-\!\!\bullet$

$$\Rightarrow \int_{\bullet\!-\!\bullet}(M_0, M_1, A) \overset{!}{=} M_0 \underset{A}{\otimes} M_1 .$$

∎

Lecture VIII

Spectra and invariants of Legendrians

The fourth and final lecture of week two introduced the notion of spectra, and touched on in-progress joint work with Lisa Traynor on defining stable homotopy invariants of Legendrians in jet bundles.

We note that the meaning of the term “spectral invariant” is now multivalued in symplectic geometry. One meaning concerns quantitative invariants (reminiscent of the eigenvalue spectrum of a linear operator) detectable through, say, persistence techniques. The meaning here instead concerns the homotopy-theoretic use of the the term spectrum, as a stable homotopy type.

We present no exercises for this lecture. For further reading and for exercises on spectra, we refer the reader to the Vancouver lectures [16].

For the homological precursors to these spectral invariants, see the works [21, 12, 3]. The key geometric computations regarding the fibration property are contained in [13, 20].

2023-8-10(木) HA in G ⑧ (FINAL)

Option A : Intro. to spectra
+
Stable htpy inv.s
for Legendrian
in jet bundles

jt w/ L. Traynor (in progress)

Option B : Morse thy on a point,
a stack of broken lines,
and ass. algebras.

jt w/ J. Lurie · on the arXiv
· minimal exposition
regarding ∞-cat.s,
and their use.

Spectra

Q. If you were a topologist, how would you define a topological / htpy-theoritic notion of "abelian group"?

E.g. ①

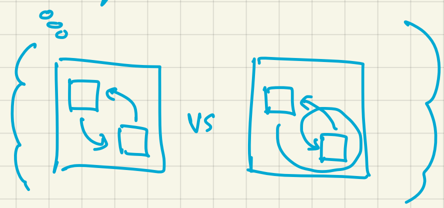
X : top. sp. , $x_0 \in X$

$$\pi_1(X, x_0) \overset{\text{def.}}{=} \left\{ ([0,1], \partial) \xrightarrow{\gamma} (X, x_0) \right\} \Big/ \text{htpy rel } \partial .$$

: NOT abelian in general.

$$\pi_2(X, x_0) \overset{\text{def.}}{=} \left\{ ([0,1]^2, \partial) \xrightarrow{\gamma} (X, x_0) \right\} \Big/ \text{htpy rel } \partial .$$

: abelian, but NOT canonically abelian!



To remember this :

Lift $\pi_2(X, x_0) \rightsquigarrow \Omega^2 X = \left\{ ([0,1]^2, \partial) \xrightarrow{\gamma} (X, x_0) \right\}$

(More generally, $\Omega^n X = \left\{ ([0,1]^n, \partial) \xrightarrow{\gamma} (X, x_0) \right\}$ remembers more than π_n.)

The obstruction to $\Omega^n X$ being "commutative" was encoded in S^{n-1}.

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \cdots \hookrightarrow S^\infty$$ contractible! ($\Rightarrow$ abelian)

<u>Upshot</u> : If we can give a space X the structure of

$$\Omega^n X_n \quad \left(\begin{array}{l}\text{for some} \\ \text{pt.d space } X_n\end{array}\right)$$

$\forall n$, in a compatible way,

X deserves to be called "abelian".

<u>Def.</u> A spectrum is the data of :

- pt.d spaces $X_0, X_1, X_2, \ldots$
- $\forall n \geq 0$, a htpy equiv. $X_n \to \Omega X_{n+1}$.

$\rightsquigarrow \quad X_0 \xrightarrow{\sim} \Omega X_1$ ← looks like a group!

Given $X_1 \underset{\sim}{\xrightarrow{f_1}} \Omega X_2$, consider $\Omega X_1 \underset{\sim}{\xrightarrow{\Omega f_1}} \Omega^2 X_2$.

Then we have $X_0 \xrightarrow{\sim} \Omega X_1 \xrightarrow{\sim} \Omega^2 X_2$,

and $\forall n$, $X_0 \xrightarrow{\sim} \Omega^n X_n$. $\Rightarrow$ $\left\{\begin{array}{l}X \text{ is given} \\ \text{a multiplication} \\ \text{as comm. as we like!}\end{array}\right\}$

Def. A map of spectra $j : X \to Y$ is the data of C^0 maps $\{ j_n : X_n \to Y_n \}_{n \geq 0}$ w/ htpys H_n :

$$\begin{tikzcd} X_n \arrow[r, "j_n"] \arrow[d, "s"'] \arrow[dr, phantom, "H_n"] & Y_n \arrow[d, "s"] \\ \Omega X_{n+1} \arrow[r, "\Omega j_{n+1}"'] & \Omega Y_{n+1} \end{tikzcd}.$$

E.g. Fix an abelian group A.

(Thm. :) $\forall n \geq 0$, $\exists K(A,n)$: top. sp.

s.t. $\pi_i K(A,n) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$

Eilenberg-MacLane sp.

Take $X_n := K(A,n)$.

$$\rightsquigarrow \quad \begin{tikzcd} X_n \arrow[r] \arrow[d, equal] & \Omega X_{n+1} \arrow[d, equal] \\ K(A,n) \arrow[dr, dotted, "!"'] & \Omega K(A,n+1) \arrow[d, "\wr"] \\ & K(A,n) \end{tikzcd}$$

This spectrum is called HA.

Eilenberg–MacLane spectrum

<u>E.g.</u> Define

$$X_0 := \operatorname{colim}\left(\mathbb{S}^0 \to \underset{\Omega \mathbb{S}^1}{\overset{\|}{\Omega \Sigma \mathbb{S}^0}} \to \underset{\Omega^2 \mathbb{S}^2}{\overset{\|}{\Omega^2 \Sigma^2 \mathbb{S}^0}} \to \cdots \right) = \bigcup_{n \geq 0} \Omega^n \mathbb{S}^n,$$

and set $X_k := \bigcup_{n \geq k} \Omega^{n-k} \mathbb{S}^n$.

$$\rightsquigarrow \quad \Omega^k X_k = \bigcup_{n \geq k} \Omega^k \Omega^{n-k} \mathbb{S}^n = \bigcup_{n \geq k} \Omega^n \mathbb{S}^n = X_0.$$

This spectrum is called $\mathbb{S}$.

the sphere spectrum.

<u>Rmk.</u> Can define the ∞-cat. of spectra in two ways:

① $$\begin{array}{ccc} \lim\left(\cdots \to \mathcal{T}op_* \xrightarrow{\Omega} \mathcal{T}op_* \xrightarrow{\Omega} \mathcal{T}op_* \right) & & \\ \| & X_2 \to X_1 \to X_0 & \\ \text{Spectra.} & & \end{array}$$

$\mathcal{T}op_* := \mathcal{T}op_{pt/}$

e.g. $\Delta^0 \to \mathrm{Spectra}$

$$= \left(\begin{tikzcd} & & & \Delta^0 \arrow[dl, "X_2"'] \arrow[d, "X_1"] \arrow[dr, "X_0"] & \\ \cdots \arrow[r] & \mathcal{T}op_* \arrow[r, "\Omega"] & \mathcal{T}op_* \arrow[r, "\Omega"] & \mathcal{T}op_* & \end{tikzcd} \right)$$

gives

$\Omega X_1 \xrightarrow{\sim} X_0$!

$\Omega X_2 \xrightarrow{\sim} X_1$

$\Omega^n X_n \xrightarrow{\sim} X_0$

② $\mathrm{Spectra} \simeq \mathrm{colim}\left(\cdots \xleftarrow{\Sigma} \mathcal{T}op_* \xleftarrow{\Sigma} \mathcal{T}op_* \right)$,

because Σ is a left adj. to Ω and

$$\left\{ \begin{array}{l} \infty\text{-cat.s w/} \\ \text{maps} = \text{right adj.s} \end{array} \right\}^{op} \simeq \left\{ \begin{array}{l} \infty\text{-cat.s w/} \\ \text{maps} = \text{left adj.s} \end{array} \right\}.$$

This colimit has a model (involving 2 steps), and step 1 is explicit :

$$\left\{ \begin{array}{l} \cdot \ \text{pt.d spaces} \quad A_0, A_1, A_2, \ldots \\ \cdot \ \forall n \geq 0, \ \text{htpy equiv.} \quad A_{n+1} \leftarrow \Sigma A_n . \end{array} \right.$$

Remember

$\{A \to \Omega B\} \simeq \{\Sigma A \to B\}$

$$\begin{array}{ccc} \mathrm{Ab} & \xrightarrow{\text{EM spectrum}} & \mathrm{Spectra} \\ \otimes_{\mathbb{Z}} & \longmapsto & \otimes = \overset{\text{smash}}{\wedge} \end{array}.$$

<u>Recall</u> : We have

$$\mathrm{Sets}_* \underset{\text{Forget}_*}{\overset{\text{Free}_*}{\rightleftarrows}} \mathrm{Ab}_*$$

$$\begin{array}{ccc} A & \longmapsto & \mathbb{Z}A = \mathbb{Z}^{\oplus A}, \\ (A, a_0) & \longmapsto & \mathbb{Z}A / \mathbb{Z}\{a_0\} \\ G \text{ as set} & \longleftarrow\!\shortmid & G \text{ as ab. gp.} \\ (G, e) & \longleftarrow\!\shortmid & G \\ \times & \overset{\text{Free}}{\longmapsto} & \otimes \\ \wedge & \overset{\text{Free}_*}{\longmapsto} & \otimes \end{array}$$

and $\hom_{\mathrm{Ab}}(\mathbb{Z}A, G) \underset{1:1}{\overset{\text{nat.}}{\longleftrightarrow}} \hom_{\mathrm{Sets}}(A, G)$.

<u>Thm.</u> $\exists$ a "free-forget" adjunction

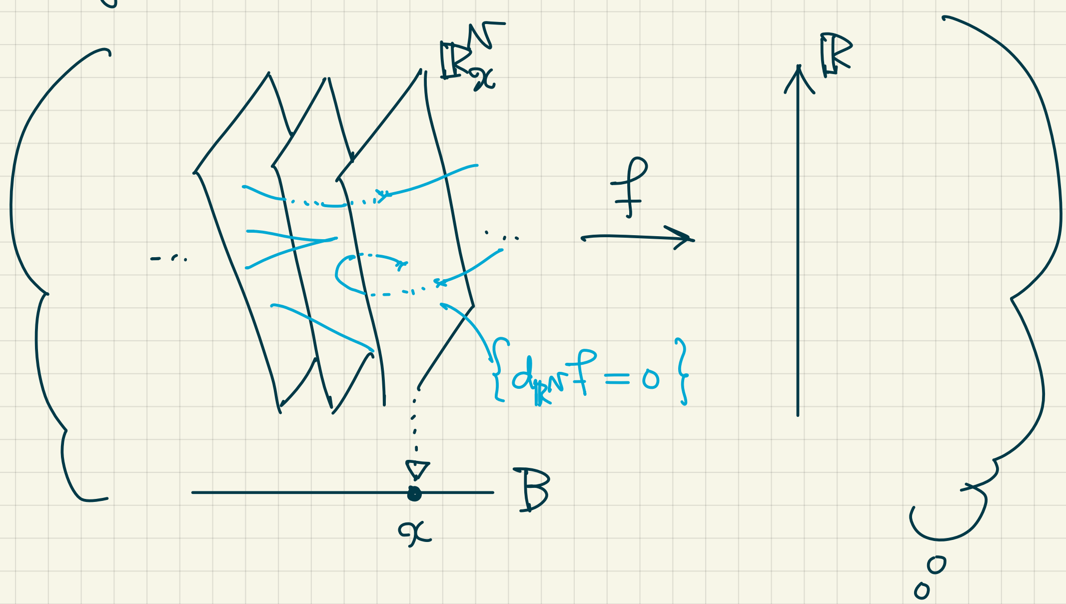
$$\mathrm{Top}_* \rightleftarrows \mathrm{Spectra}$$

$$\begin{array}{ccc} A & \longmapsto & \text{(say) } \Sigma^\infty A \\ X_0 & \longleftarrow\!\shortmid & X \\ \wedge & \longmapsto & \otimes \end{array}$$

※ This proof is NOT elementary.

$$\begin{tikzcd}
\mathrm{Top}_* \arrow[r, shift left, "1"] \arrow[d, "\pi_0"'] & \mathrm{Spectra} \arrow[l, shift left] \arrow[d, "\text{``}\pi_0\text{''}"] \\
\mathrm{Sets}_* \arrow[r, shift left] & \mathrm{Ab} \arrow[l, shift left, "\text{famous}"]
\end{tikzcd}$$

Legendrians :



Def. Fix B C^∞ mfd, and

$$f : B \times \mathbb{R}^N \to \mathbb{R}.$$

Consider the locus

$$\left(x,\ d_B f(x,\eta),\ \eta,\ f(x,\eta)\right)$$

for those $(x,\eta) \in B \times \mathbb{R}^N$ where $d_{\mathbb{R}^N} f = 0$.

remember values of f

$$\begin{tikzcd}
 & & & \Lambda_f \arrow[dlll, dashed] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{p.b.}"] & & & \\
T^*B \times \mathbb{R} \arrow[dr] & & \mathrm{Graph}(df) \arrow[dl] \arrow[dr] & & \mathbb{R}^N \arrow[dl, "\text{zero section}"'] \arrow[dr] & & \\
 & T^*B & & T^*\mathbb{R}^N & & \mathrm{pt}. &
\end{tikzcd}$$

Prop. (Exc.) Λ_f is a Legendrian (immersed) in $T^*B \times \mathbb{R}$. $= J^1(B, \mathbb{R})$

$\star$ f is called a generating family for Λ_f.

Cnstrxn (L. Traynor)

Given a generating family f for Λ, consider the C^∞ fxn

assume Λ is embedded.

$$\begin{aligned} B \times \mathbb{R}^N \times \mathbb{R}^N &\xrightarrow{\Delta_f} \mathbb{R} \\ (x, \eta, \tilde{\eta}) &\longmapsto f(x,\eta) - f(x,\tilde{\eta}). \end{aligned}$$

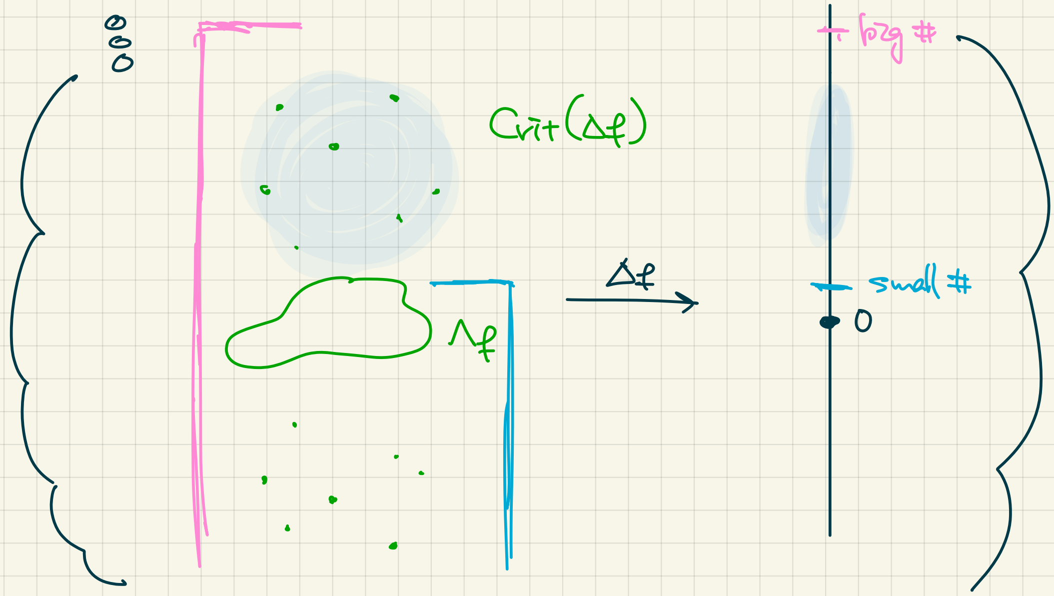
Prop.

$$\underset{B\times\mathbb{R}^N\times\mathbb{R}^N}{\operatorname{Crit}(\Delta_f)} \cong \Lambda_f \cup \left\{ \begin{matrix} \text{Reeb chords} \\ \Lambda_f \text{ to itself} \end{matrix} \right\}^{\amalg 2}$$

where $\Delta_f = 0$

where $\Delta_f > 0$
< 0.

$\star$ $A(f) := \dfrac{\{\Delta_f \leq \text{big } \#(>0)\}}{\{\Delta_f \leq \text{small } \#(>0)\}} \in \mathcal{T}op_*$.



Lemma (Chekanov)

$$\begin{array}{ccccc} \forall f & \in & \mathcal{G}en_N & \xrightarrow[\text{stabilization by } +x_1^2+\cdots+x_K^2]{} & \mathcal{G}en_{N+K} \\ \downarrow & & \downarrow & \circlearrowright & \swarrow \\ \Lambda_f & \in & \mathcal{L}og & & \end{array}$$

$\forall \gamma$: Leg. isot.

Λ'

depends on γ

$\exists K \gg 1$ s.t. γ lifts to $\mathcal{G}en_{N+K}$.

Thm. (Cor.)

$$\mathcal{G}en := \bigcup_{N \geq 0} \mathcal{G}en_N \longrightarrow \mathcal{L}og \quad \text{is a fibration.}$$

Lemma

$$f \longmapsto f + x^2$$

$$\begin{tikzcd}
\mathcal{G}en_N \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{htpy}"] & \mathcal{G}en_{N+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{htpy}"] & \mathcal{G}en_{N+2} \arrow[r] \arrow[d] & \cdots \\
\mathcal{T}op_* \arrow[r] & \mathcal{T}op_* \arrow[r] & \mathcal{T}op_* \arrow[r] & \cdots
\end{tikzcd}$$

$$A(f) \longmapsto \Sigma A(f)$$

Cor. We have a functor

$$\mathcal{G}en \longrightarrow \text{Spectra}.$$

$$(\Lambda f, f)$$

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