Reading 25

Closures, boundaries, and density

25.1 Closure

Definition 25.1.1. Fix a topological space X and let $B \subset X$ be a subset.[1] Let

$$\mathcal{K}_B$$

be the collection of all closed subsets of X containing B.[2] Then the *closure* of B is defined to be

$$\overline{B} := \bigcap_{K \in \mathcal{K}_B} K.$$

In words, the closure of B is the set obtained by intersecting every closed subset containing B.

Remark 25.1.2. Note that B is always a subset of $\overline{B}$.

Remark 25.1.3. Note that $\overline{B}$ is a closed subset of X. This is because the intersection of closed subsets is always closed.

Example 25.1.4. Let U be an open subset containing x. Then the closure $\overline{U}$ is a neighborhood of x.

[1] It could be any kind of subset: open, closed, neither!

[2] Note that X is an element of $\mathcal{K}_B$.

Remark 25.1.5. If $B \subset X$ is closed, then $\overline{B} = B$. To see this, note that B is an element of $\mathcal{K}$ because B is closed. Hence

$$\bigcap_{K \in \mathcal{K}} K = B \cap \left(\bigcap_{K \in \mathcal{K}, K \neq B} K \right).$$

But this righthand side is a subset of B because it is obtained by intersecting B with some other set. In particular,

$$\overline{B} \subset B.$$

Because $B \subset \overline{B}$ (for any kind of B), we conclude that $B = \overline{B}$.

The converse is also true: If $\overline{B} = B$, then B is closed.

Example 25.1.6. If $B = \emptyset$, then $\overline{B} = \emptyset$. If $B = X$, then $\overline{B} = X$.

25.2 Density

Definition 25.2.1. Let X be a topological space and fix a subset $B \subset X$. We say that B is *dense* in X if $\overline{B} = X$.

25.3 Exercises about closures

Exercise 25.3.1. Let $X = \mathbb{R}^n$ (with the standard topology). Let $B = \mathrm{Ball}(0, r)$ be the open ball of radius r. Show that the closure of B is the closed ball of radius r; that is,

$$\overline{B} = \{x \in \mathbb{R}^n \text{ such that } d(x, 0) \leq r\ .\}$$

Proof. You showed in your homework that if $K \subset X$ is closed and if $x_1, \ldots$ is a sequence in K converging to some $x \in X$, then x is in fact an element of K.

Choose a point x of distance r from the origin. And choose also an increasing sequence of positive real numbers $t_1, t_2, \ldots$ converging to 1.[3] Then the sequence

$$x_i = t_i x$$

[3] For example, you could take $t_i = i/(i+1)$.

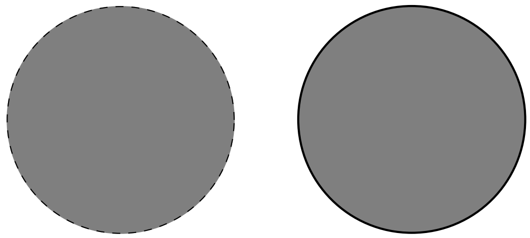

Figure 25.1: An open ball on the right; its closure (a closed ball) on the left.

is a sequence in B converging to x. If $K \supset B$, then the x_i define a sequence in K; moreover, if K is closed, the limit x is in K. Thus $x \in K$ for any closed subset containing B. In particular, x is in the intersection of all such K. Thus $x \in \overline{B}$. This shows that the closed ball of radius r is contained in $\overline{B}$.

On the other hand, consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $d(0, -)$; that is, the "distance to the origin" function. We see that $f^{-1}([0, r])$ is equal to the closed ball of radius r—in particular, this closed ball is a closed subset of $\mathbb{R}^n$, and it obviously contains $\mathrm{Ball}(0, r)$. This shows that $\overline{B}$ is a subset of the closed ball of radius r (because $\overline{B}$ can be expressed as the intersection of this closed ball with other sets). We are finished. $\square$

Exercise 25.3.2. Suppose $f : X \to Y$ is a continuous function, and let $B \subset X$ be a subset. Show that

$$f(\overline{B}) \subset \overline{f(B)}.$$

In English: The image of the closure of B is contained in the closure of the image of B.

Proof. Let $\mathcal{C}$ be the collection of closed subsets of Y containing $f(B)$. Then

$$f^{-1}(\overline{f(B)}) = f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right)$$

by definition of closure. We further have:

$$f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right) = \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

Now, because f is continuous, we know that $f^{-1}(C)$ is closed for every $C \in \mathcal{C}$. Moreover, because $f(B) \subset C$, we see that $B \subset f^{-1}(C)$. We conclude that for every $C \in \mathcal{C}$, $f^{-1}(C) \in \mathcal{K}$. Thus

$$\bigcap_{K \in \mathcal{K}} K \subset \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

The lefthand side is the definition of $\overline{B}$. The righthand side is $f^{-1}(\overline{f(B)})$. We are finished. □

Remark 25.3.3. It is not always true that $f(\overline{B})$ is equal to $\overline{f(B)}$. For example, let $B = X = \mathrm{Ball}(0, r)$, and let $f : X \to \mathbb{R}^2$ be the inclusion. Then $f(\overline{B}) = X$, while $\overline{f(B)}$ is the closed ball of radius r.

Exercise 25.3.4. Find an example of a continuous function $p : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{\{x \text{ such that } p(x) < t\}},$$

does *not* equal

$$\{x \text{ such that } p(x) \leq t\}.$$

Example 25.3.5. Let $B \subset \mathbb{R}^2$ be the following subset:

$$B = \{(x_1, x_2) \text{ such that } x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\} \subset \mathbb{R}^2.$$

This is not a closed subset of $\mathbb{R}^2$. I claim

$$\overline{B} = B \bigcup \{(x_1, x_2) \text{ such that } x_1 = 0 \text{ and } x_2 \in [-1, 1]\}.$$

That is, $\overline{B}$ is equal to the so-called topologist's sine curve.

Let us call the righthand side S for the time being. First, I claim that $S \subset \overline{B}$. Indeed, fix some point $(0, T) \in S \setminus B$. Then there is an unbounded, increasing sequence of real numbers $t_1, t_2, \ldots$ for which $\sin(t_i) = T$; let $s_i = 1/t_i$. Then the sequence of points

$$x_i = (s_i, \sin(1/s_i)) = (s_i, T)$$

converges to $(0, T)$, while each x_i is an element of B. In particular, $(0, T)$ is contained in any closed subset containing B. This shows $S \subset \overline{B}$.

To complete the proof, it suffices to show that S is closed. For this, because $\mathbb{R}^2$ is a metric space, it suffices to show that any convergent sequence contained in S has a limit contained in S. So let $x_1, x_2, \ldots$ be a sequence in S.

Suppose that the limit $x \in \mathbb{R}^2$ has the property that the 1st coordinate is non-zero. There is a unique point in S with a given non-zero first coordinate t, namely $(t, \sin(1/t))$. Moreover, because the function $t \mapsto \sin(t/1)$ is continuous, if $t_i = \pi_1(x_i)$ converges to t, we know that $(t_i, \sin(1/t_i))$ converges to $(t, \sin(1/t))$. So the limit is in S.

If on the other hand the first coordinate of x is equal to zero, let us examine the second coordinates $\pi_2(x_1), \ldots$. By continuity of π_2, the sequence $\pi_2(x_1), \pi_2(x_2), \ldots$ converges to some T; because each x_i has a second coordinate in $[-1, 1]$, and because $[-1, 1] \subset RR$ is closed, we conclude that the limit T is also contained in $[-1, 1]$. Hence the limit of the sequence $x_1, \ldots,$ is the point $(0, T)$, and $(0, T) \in S$.

Because any sequence in S with a limit in $\mathbb{R}^2$ has limit in S, S is closed.

25.4 Density exercises

Exercise 25.4.1. For each of the following examples of subsets of $\mathbb{R}^2$, identify the closure, the interior, and the boundary. Which of these is dense?

1. $B = \{(x_1, x_2) \text{ such that } x_1 \neq 0 \}$.

2. $B = \bigcup_{(a,b) \in \mathbb{Z} \times \mathbb{Z}} (a - 1, a + 1) \times (b - 1, b + 1)$.

3. $B = \{(x_1, x_2) \text{ such that at least one of the coordinates is rational}\}$.

Exercise 25.4.2. Prove each of the following propositions.

Proposition 25.4.3. Fix $B \subset X$. The following are equivalent:

1. B is dense in X.

2. For every non-empty open $U \subset X$, $U \cap B \neq \emptyset$.

3. For every $x \in X$, and every neighborhood A of x in X, we have that $A \cap B \neq \emptyset$.

4. For every $x \in X$, and every open neighborhood A of x in X, we have that $A \cap B \neq \emptyset$.

Proposition 25.4.4. $\mathbb{Q} \subset \mathbb{R}$ is dense.

Proposition 25.4.5. $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Solutions to Lecture 25 Propositions

Proof of Proposition 25.4.3. $1 \implies 2$. Proof by contrapositive. Suppose that there is some non-empty open $U \subset X$ such that $U \cap B = \emptyset$. Then U^C is closed while $U^C \supset B$, so the closure of B is contained in U^C by definition of closure. In particular, $\overline{B}$ does not contain U, so could not equal all of X.

$2 \implies 4$. This is obvious, as if A is an open neighborhood of x, then A is a non-empty open subset of X.

$4 \implies 3$. Given A a neighborhood of x, let $U \subset A$ be the open subset containing x (guaranteed by the definition of neighborhood). Then $U \cap B \neq \emptyset$ by 4, so $A \cap B \supset U \cap B \neq \emptyset$.

$3 \implies 1$. Clearly $\overline{B} \subset X$ always, so we must show that $X \subset \overline{B}$. Let $K \subset X$ be a closed subset containing B. Then K^C is open. If K^C is non-empty, choose $x \in K^C$, and note that K^C is a neighborhood of x. Thus by 3, $K^C \cap B \neq \emptyset$; this contradicts the fact that $B \subset K$. □

Proof of Proposition 25.4.4. Let $x \in \mathbb{R}$ be a real number, and for every integer $n \geq 1$, let x_n be any rational number in the interval $(x - 1/n, x + 1/n)$. Then the sequence x_n converges to x. By the sequence criterion for closure, we thus see that any real number is in the closure of $\mathbb{Q}$. □

Proof of Proposition 25.4.5. Same exact proof, except choose each x_n to be any *irrational* number in the interval $(x - 1/n, x + 1/n)$. □