Reading 24

Interiors and neighborhoods

24.1 Interiors

Definition 24.1.1. Let X be a topological space and fix $B \subset X$. Let $\mathcal{U}_B$ denote the collection of open subsets of X that are contained in B. Then the *interior* of B is defined to be the union

$$int(B) = \bigcup_{U \in \mathcal{U}_B} U.$$

Remark 24.1.2. Informally, the interior of B is the largest open subset of X that is contained in B.

Remark 24.1.3. For any B, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both B and of X.

Remark 24.1.4. If B is open, then $int(B) = B$. This is because $B \in \mathcal{U}_B$, so

$$int(B) = \bigcup_{U \in \mathcal{U}_B} U = B \cup \left(\bigcup_{U \neq B, U \in \mathcal{U}_B} U \right)$$

meaning $int(B)$ contains B (because $int(B)$ is a union of B with possibly other sets). Thus we have that $int(B) \subset B \subset int(B)$, meaning $int(B) = B$.

Example 24.1.5. We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

Example 24.1.6. Let $X = \mathbb{R}^n$ and let B be the closed ball of radius r about the origin. Then $int(B) = \text{Ball}(0, r)$ is the open ball of radius r.

To see this, we note that $\text{Ball}(0, r)$ is open and contained in B, so $\text{Ball}(0, r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of B (i.e., no point in $B \setminus \text{Ball}(0, r)$) is contained in the interior of B.

So fix $y \in B \setminus \text{Ball}(0, r)$, meaning y is a point of exactly distance r away from the origin. It suffices to show that there is no open ball containing y and contained in B; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\text{Ball}(y, \delta) \subset \mathbb{R}^n$ contains some point of distance $> r$ from the origin. So $\text{Ball}(y, \delta)$ is never contained in B. This completes the proof.

24.2 Neighborhoods

Definition 24.2.1. Let X be a space and let x an element of X. A subset $A \subset X$ is called a *neighborhood* of x if there exists some open $U \subset X$ with $x \in U$ for which $U \subset A$.

When A is also open, we say A is an *open neighborhood* of x.

Remark 24.2.2. In topology, we use the word "neighborhood" usually when we're being lazy. "Neighborhood of x" is shorter than saying "subset containing an open subset containing x."

Note that a neighborhood of x need not be an open subset.

Here is one connection between the ideas of interiors and neighborhoods. Let X be a space. Fix a subset $A \subset X$ and an element $x \in X$.

Proposition 24.2.3. *A is a neighborhood of x if and only if the interior of A contains x.*

24.3 Exercises on interiors

Exercise 24.3.1. Prove Proposition 24.2.3.

Exercise 24.3.2. Let's compute the interiors of some common examples.

(a) Show that the interior of S^{n-1} (as a subset of $\mathbb{R}^n$) is empty.

(b) Show that the interior of S^{n-1} – as a subset of itself – is S^{n-1}.

(c) Show that the interior of Δ^{n-1} (as a subset of $\mathbb{R}^n$) is empty.

(d) Show that the interior of Δ^{n-1} – as a subset of itself – is Δ^{n-1}.

(e) Show that the interior of $\mathbb{Q}$ (as a subset of $\mathbb{R}$) is empty.

Exercise 24.3.3. Let $U \subset X$ be an open subset. Show that $int(U) = U$.

Exercise 24.3.4. Let A and B be subsets of X.

(a) Show that $int(A \cup B) \supset int(A) \cup int(B)$.

(b) Show by example that $int(A \cup B)$ is not equal to $int(A) \cup int(B)$.

Exercise 24.3.5. Let X be a metric space (endowed with the metric topology). Show that for any subset A of X, $int(A)$ is the union of all open balls contained in A.

Exercise 24.3.6. Let A and B be subsets of X.

(a) Show that $int(A \cap B) = int(A) \cap int(B)$.

(b) Let $\{B_\alpha\}_{\alpha \in \mathcal{A}}$ be a possibly infinite collection of subsets of X. Show that $int(\bigcap_{\alpha \in \mathcal{A}} B_\alpha) \subset \bigcap_{\alpha \in \mathcal{A}} int(B_\alpha)$.

(c) Show by example that $int(\bigcap_{\alpha \in \mathcal{A}} B_\alpha)$ need not equal $\bigcap_{\alpha \in \mathcal{A}} int(B_\alpha)$.

24.4 Exercises on neighborhoods

Exercise 24.4.1. Let $X = \mathbb{R}^2$ and let x be the origin. For each description below, find an example of $B \subset \mathbb{R}^2$ fitting the description:

(a) B is an open neighborhood of x. (This means B is a neighborhood of x, and is an open subset.)

(b) B is a closed neighborhood of x. (This means B is a neighborhood of x, and is a closed subset.)

(c) B is a closed and open neighborhood of x.

(d) B is a closed but not open neighborhood of x.

(e) B is an open but not closed neighborhood of x.

(f) B is a neither open nor closed neighborhood of x.

(g) B is a compact, and not open, neighborhood of x.

Exercise 24.4.2. Let X be a topological space and fix $x \in X$.

(a) Let $B_1, \dots, B_k$ be a finite collection of neighborhoods of x. Show that $B_1 \cap \dots \cap B_k$ is a neighborhood of x.

(b) Show by example that the intersection of infinitely many neighborhoods of x need not be a neighborhood.

Exercise 24.4.3. Suppose $f : X \to Y$ is a function. Show that the following conditions are equivalent.

(a) f is continuous.

(b) For all $x \in X$, if N is a neighborhood of $f(x)$, then the preimage of N is a neighborhood of x.

(c) For all $x \in X$, if N is a neighborhood of $f(x)$, there exists a neighborhood of x contained in $f^{-1}(N)$.

(d) For all $x \in X$, if N is an open neighborhood of $f(x)$, there exists an open neighborhood of x contained in $f^{-1}(N)$.

(e) For all $x \in X$, if N is an open neighborhood of $f(x)$, then the preimage of N is an open neighborhood of x.

Exercise 24.4.4. Suppose $f : X \to Y$ is a function and fix $x \in X$. Show that the following conditions are equivalent.

(a) (Omitted.)

(b) If N is a neighborhood of $f(x)$, then the preimage of N is a neighborhood of x.

(c) If N is a neighborhood of $f(x)$, there exists a neighborhood of x contained in $f^{-1}(N)$.

(d) If N is an open neighborhood of $f(x)$, there exists an open neighborhood of x contained in $f^{-1}(N)$.

Remark: Sometimes, the equivalent conditions above are the definition of f being continuous *at* x.

Show by example that the following is *not* equivalent to the others:

(e) If N is an open neighborhood of $f(x)$, then the preimage of N is an open neighborhood of x.

Make sure you contrast this exercise with Exercise 24.4.4.

Possible solution to Exercise 24.4.1. (a) $\mathbb{R}^2$ is an open neighborhood of x.

(b) $\mathbb{R}^2$ is a closed neighborhood of x.

(c) $\mathbb{R}^2$ is closed and open neighborhood of x.

(d) D^2 is a closed neighborhood of x that is not open.

(e) $\text{Ball}(0, r)$ for any $r > 0$ is an open neighborhood of x that is not closed.

(f) Consider the set $B = D^2 \setminus \{((\sqrt{2})/2, (\sqrt{2})/2)\}$ – in fact, we could remove any point from S^1. This is a neighborhood of x that is neither open nor closed.

(g) D^2 is a compact and not open neighborhood of x.

□

Possible solution to Exercise 24.4.3. (a) $\implies$ (b). If N is a neighborhood of $f(x)$, there exists an open V for which $f(x) \in V \subset N$. Since f is continuous $f^{-1}(V)$ is open, while by straightforward arguments using sets we have $x \in f^{-1}(V) \subset f^{-1}(N)$.

(b) $\implies$ (c). If the preimage of N is a neighborhood of x, note that $f^{-1}(N) \subset f^{-1}(N)$ – so the preimage itself is a neighborhood of x contained in $f^{-1}(N)$.

(c) $\implies$ (d). Since (c) holds for all neighborhoods, it holds for open neighborhoods. If M is the neighborhood of x guaranteed by (c), there exists an open subset U satisfying $x \in U \subset M \subset f^{-1}(N)$ by definition of neighborhood. This U is an open neighborhood of x contained in $f^{-1}(N)$.

(d) $\implies$ (a). Suppose $V \subset Y$ is open and let $U = f^{-1}(V)$. If U is empty, it is open. Otherwise, let $x \in U$. Then V is an open neighborhood of $f(x)$

– it follows from (d) that $f^{-1}(V) = U$ is an open neighborhood of x, and is hence is open. □

Possible solution to Exercise 24.4.4. (b) $\implies$ (c). If the preimage of N is a neighborhood of x, note that $f^{-1}(N) \subset f^{-1}(N)$ – so the preimage itself is a neighborhood of x contained in $f^{-1}(N)$.

(c) $\implies$ (d). Since (c) holds for all neighborhoods, it holds for open neighborhoods. If M is the neighborhood of x guaranteed by (c), there exists an open subset U satisfying $x \in U \subset M \subset f^{-1}(N)$ by definition of neighborhood. This U is an open neighborhood of x contained in $f^{-1}(N)$.

(d) $\implies$ (b). If N is a neighborhood of $f(x)$, there is some open subset V satisfying $f(x) \in V \subset N$. Since V is an open neighborhood of $f(x)$, (d) guarantees the existence of an open set U satisfying $x \in U \subset f^{-1}(V)$. Because $U \subset f^{-1}(N)$, (a) follows.

Now, clearly (e) $\implies$ (d). But the converse is not true. As an example, consider a function $f : \mathbb{R} \to [1]$ with

$$f(x) = \begin{cases} 1 & x \geq \pi \\ 0 & \text{otherwise.} \end{cases}$$

Give the domain the standard topology and the codomain the poset topology. Then if $x > \pi$, we see that $\{1\} \subset [1]$ is an open neighborhood of $f(x)$; but the preimage of $[1]$ is precisely the interval $[\pi, \infty)$ – which is not open. On the other hand, because $x > \pi$, there does exist an open neighborhood of x fully contained in the preimage of $\{1\}$. This shows that (d) does not imply (e). □