Reading 23

One-point compactification

23.1 Stereographic projection

Stereographic projection is the function

$$p : S^n \setminus \{(0, \ldots, 0, 1)\} \to \mathbb{R}^n, \qquad (x_1, \ldots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

Here is a description of p in words. For brevity, let us call the point $(0, \ldots, 0, 1) \in S^n$ the *north pole* of S^n. Given a point $x \in S^n$ such that x is not the north pole, p sends x to the intersection of

- the line through x and the north pole, with
- the hyperplane $\{x_{n+1} = 0\}$, which one can identify with $\mathbb{R}^n$.

See Figure 23.1.

Note that the domain of stereographic projection is not all of S^n, but S^n minus the north pole. Notice also that p is a bijection; this gives us an informal way to think about S^n—it is obtained from $\mathbb{R}^n$ by "adding one point" that plays the role of the north pole of S^n.

Remark 23.1.1. Here is one way to think about this: Imagine a nice smooth rubber ball. If you puncture the rubber ball in one place (say, with a needle), you can actually stretch out the entire rubber ball onto a flat surface. In fact, by stretching and stretching, you can cover the entire plane.

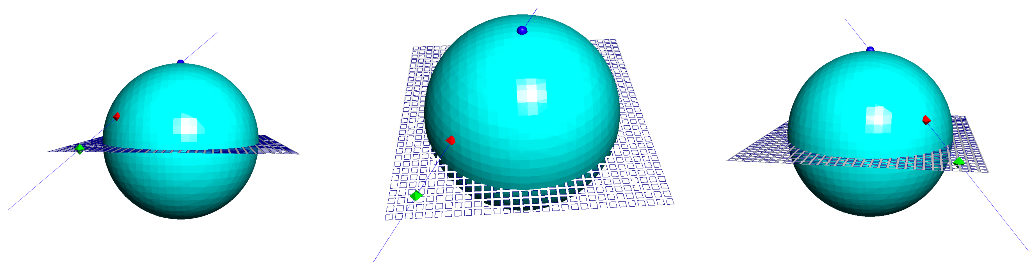

Figure 23.1: A depiction of stereographic projection for S^2. In blue is the north pole, from three different angles; the red is an element $x \in S^2$, and in green is the image of x under stereographic projection. Drawn also are S^2, the plane $\mathbb{R}^2$ (embedded as the subset where $x_3 = 0$) and the line connecting x to its image.

And, by adding this one point to $\mathbb{R}^n$, we obtain a compact topological space (the sphere). It is important here that we know how to *topologize* this set obtained by adjoining a point to $\mathbb{R}^n$. (There are ways to topologize this set that do not result in S^n, for example.) There is a particularly nice process of adjoining one point to a space, to obtain a new, compact space – *one-point compactification*.

23.2 One-point compactification

Let X be a topological space. We are now going to create a new topological space X^+.

Definition 23.2.1. Given a set X, let $X^+ = X \coprod \{*\}$.[1] We endow X^+ with a topology $\mathcal{T}_{X^+}$ defined as follows: $U \subset X^+$ is open if either

1. $* \notin U$ and U is open in X, or

[1] In other words, X^+ is the set obtained by adjoining a single point called $*$ to X.

2. $* \in U$ and $U \cap X$ is the complement of a closed, compact subspace of X.

We call X^+ the *one-point compactification* of X.

Remark 23.2.2. Note that if X is Hausdorff, we may remove the adjective "closed" from the second condition above. (Every compact subspace of a Hausdorff space is closed.)

Remark 23.2.3. Sometimes, X^+ is called the *Alexandroff extension* of X, or the *Alexandroff compactification* of X.

23.3 Basic properties

Proposition 23.3.1. $\mathcal{T}_{X^+}$ is a topology on the set X^+.

Proposition 23.3.2. X^+ is compact.

Remark 23.3.3. Proposition 23.3.2 justifies the word "compactification."

Proposition 23.3.4. If X is compact, then X^+ is homeomorphic to the space $X \coprod \{*\}$ with the coproduct topology. (The coproduct topology on the union $X \coprod Y$ is the topology where a subset $U \subset X \coprod Y$ is open if and only if its intersection with X is open and its intersection with Y is open).

Proposition 23.3.5. If $X = \mathbb{R}^n$, then X^+ is homeomorphic to S^n.

Proposition 23.3.6. If X is Hausdorff and if every point $x \in X$ admits an open U and compact K with $x \in U \subset K$, then X^+ is Hausdorff.

Proposition 23.3.7. If X and Y are homeomorphic, so are X^+ and Y^+.

23.4 Examples/Exercises

Exercise 23.4.1. Prove Proposition 23.3.1. (You will want to use at some point that the empty set is a compact space.)

Exercise 23.4.2. Let $X = *$ be the one-element topological space. Write out the topology of the one-point compactification X^+.

Exercise 23.4.3. Let X be a discrete topological space. Is the topology of the one-point compactification X^+ also discrete?

Exercise 23.4.4. Let $\iota : X \to X^+$ denote the inclusion – it sends x to x.

(a) Show that ι is continuous.

(b) Prove ι has the property that the *image* of any open subset is open.[2]

Exercise 23.4.5. Prove Proposition 23.3.2.

Exercise 23.4.6. Prove Proposition 23.3.7.

Exercise 23.4.7. Prove Proposition 23.3.4.

Exercise 23.4.8. Prove Proposition 23.3.6.

Exercise 23.4.9. Given a function $f : X \to Y$, we define the function $f^+ : X^+ \to Y^+$ to act by $x \mapsto f(x)$ and $* \mapsto *$.

(a) Suppose $f : X \to Y$ is a continuous function. Show by example that f^+ is not necessarily continuous.

(b) Show that if f is both continuous, and has the property that f^{-1} of a compact subset is compact[3], then f^+ is continuous.

(c) Let $f : X \to Y$ and $g : Y \to Z$ be arbitrary functions. Prove $(g \circ f)^+ = g^+ \circ f^+$.

(d) Prove that any homeomorphism is proper.

(e) For any set X, let $\mathrm{id}_X : X \to X$ denote the identity function. Prove that $(\mathrm{id}_X)^+ = \mathrm{id}_{X^+}$.

(f) Prove Proposition 23.3.5.

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[2]Such a continuous map is called an *open* map.

[3]This property of f is called *properness*. That is, f is *proper* if preimages of compacts are compact.

Some possible solutions

Proof of Proposition 23.3.1. (i) We first show $\emptyset, X^+$ is in this topology. So let $U = \emptyset$. Then $* \notin U$, so we must check whether $\emptyset$ is open in X (by condition 1 of the definition of $\mathcal{T}_{X^+}$). It is, by definition of topological space (i.e., because X itself is a topological space). Now let $U = X^+$. Since $* \in U$, we must check whether $U \cap X$ is the complement of a closed, compact subspace of X (by condition 2 of the definition of $\mathcal{T}_{X^+}$). It is, because $U \cap X = X$ and X is the complement of $\emptyset$. (Note that $\emptyset$ is both closed and compact.)

(ii) Now let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an arbitrary collection where $U_\alpha \in \mathcal{T}_{X^+}$ for any $\alpha \in \mathcal{A}$. We must show that the union

$$U := \bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X^+$$

is in $\mathcal{T}_{X^+}$.

Note that for any $\alpha \in \mathcal{A}$, we know that

$$U_\alpha \cap X$$

has a complement given by a closed subspace of X. (This is true regardless of whether U_α satisfies case 1. or in case 2. of the definition of $\mathcal{T}_{X^+}$.) Let us call this closed subspace K_α, and let us call the intersection

$$K := \bigcap_{\alpha \in \mathcal{A}} K_\alpha.$$

Note that the arbitrary intersection of closed subsets is closed, so $K \subset X$ is closed. Then by de Morgan's laws, we see that

$$X \cap U = X \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) = (\bigcap_{\alpha \in \mathcal{A}} K_\alpha)^C = K^C.$$

where the complement is taken inside X. Now, if $* \notin U$, then we have shown that U^C is closed, so by condition 1. of the definition of $\mathcal{T}^+$, we see that $U \subset X^+$ is indeed in $\mathcal{T}_{X^+}$.

On the other hand, if $* \in U$, then for some $\alpha \in \mathcal{A}$, we see that $* \in U_\alpha$. In particular, K_α is not only closed, but also compact. Thus $K \subset K_\alpha$ is a closed subspace of a compact K_α, meaning K itself is compact. This shows that $U \cap X = K^C$ is the complement of a compact, closed subspace of X, so U is open by condition 2. of the definition of $\mathcal{T}_{X^+}$.

(iii) Now we must show that a finite intersection of elements in $\mathcal{T}_{X^+}$ is in $\mathcal{T}_{X^+}$.

So fix $U_1, \ldots, U_n$, a finite collection of elements in $\mathcal{T}_{X^+}$. For each i, let $K_i = (U_i \cap X)^C$. Note that K_i is closed, and is compact if $* \in U_i$. We let

$$U = U_1 \cap \ldots \cap U_n \subset X^+$$

and

$$K = K_1 \cup \ldots \cup K_n \subset X.$$

Note that by de Morgan's laws, we again have

$$U \cap X = K^C \subset X$$

(where the complement is again taken inside X).

If $* \notin U$, then $U = K^C$. Being a complement of a closed subset in X, we see that $U \subset X$ is open in X, so $U \in \mathcal{T}_{X^+}$ by condition 1. of the definition.

If $* \in U$, then $* \in U_i$ for every i, so by condition 2, each K_i is not only closed but also compact. Lemma: The finite union of compact subspaces is compact. (Proof: Given an open cover of K, note that the open cover determines a finite subcover of each K_i. Taking the union of these finite subcovers, we have a finite union of finite collections; hence the resulting union is a finite open cover of K itself.) Thus K itself is compact. By condition 2, U is in $\mathcal{T}_{X^+}$. □

Proof of Proposition 23.3.2. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X^+. By definition of cover, there is some $\alpha_0 \in \mathcal{A}$ such that $* \in U_{\alpha_0}$. So by condition 2 of the definition of $\mathcal{T}_{X^+}$, we know

$$X^+ = U_{\alpha_0} \cup K$$

where K is a compact, closed subspace of X and $K \cap U_{\alpha_0} = \emptyset$.

Before we go any further, let us point out that $X \subset X^+$ is an open subset by condition 1 of the definition of $\mathcal{T}^+$. Thus the subspace topology of $K \subset X^+$ is equal to the subspace topology of $K \subset X$.

Invoking the definition of open cover, and by definition of subspace topology (for $K \subset X$), we know that the collection

$$\{U_\alpha \cap K\}_{\alpha in \mathcal{A}}$$

is an open cover of K. Since K is compact, we can choose some finite collection $\alpha_1, \dots, \alpha_n$ so that $\{U_{\alpha_1} \cap K, \dots, U_{\alpha_n} \cap K\}$ is an open cover of K. In particular,

$$U_{\alpha_0} \cup U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

is an open cover of X^+ itself. This exhibits a finite subcover of the original open cover, and we are finished. □

Proof of Proposition 23.3.4. We must show that $W \subset X^+$ is open if and only if $W \cap X$ and $W \cap \{*\}$ is open.

To see the latter claim, we must prove that the one-element set

$$U = \{*\} \subset X^+$$

is open. This is because $U \cap X = \emptyset = X^C$, where the complement is taken in X. But X is closed (as a subset of itself), and is compact by hypothesis, so by condition 2, U is open.

On the other hand, $W \cap X$ is always open for a one-point compactification—this is obvious if $* \notin W$ by condition 1, and if $* \in W$, then $W \cap X$ is a complement of a (compact and) closed subset of X by condition 2, hence by definition of closedness, $W \cap X$ is open in X.

This completes the proof. □

Proof of Proposition 23.3.7. Given a homeomorphism $f : X \to Y$, define a function

$$g : X^+ \to Y^+, \qquad x \mapsto \begin{cases} *_Y & x = *_X \\ f(x) & x \in X. \end{cases}$$

Here, $*_Y \in Y^+$ represents the "extra point" in the one-point-compactification of Y, and likewise for $*_X \in X^+$.

Clearly g is a bijection because f is. Let us show that $U \subset X^+$ is open if and only if $g(U) \subset Y^+$ is open.

1. If $*_X \notin U$, then $*_Y \notin g(U)$. But because f is a homeomorphism, $g(U) = f(U)$ is open if and only if $U \cap X = U$ is open.

2. If $*_X \in U$, then $*_Y \in g(U)$. This means that $U \cap X = K^C$ (where the complement is taken in X) for some compact, closed $K \subset X$. But because f is a homeomorphism, $K \subset X$ is compact and closed if and only if $f(K) \subset Y$ is also compact and closed. Thus $f(U) \cap Y$ is the complement of a closed, compact subspace of Y if and only if $U \cap X$ is the complement of a closed, compact subspace of X. This completes the proof. □