Reading 22

STEP UP group exercise: Connectedness

22.1 Discussion of the statement

Recall the following definition:

Definition 22.1.1. A topological space X is called *connected* if: Given two non-empty open subsets $U, U' \subset X$, if $U \cup U' = X$, then $U \cap U' \neq \emptyset$.

Today we're going to contemplate the following proposition:

Proposition 22.1.2. Let $f : X \to Y$ be a homeomorphism. If X is connected, then Y is connected.

Discussion. Let's say we want to prove the proposition.

- What do we get to assume when proving the proposition?
- What do we need to prove?

22.2 Proof exchange

Get into your groups of two.

Each group will get two proofs: Proof A and Proof B.

1. On your own (as an individual), try to make sense of the argument you are given. Be prepared to summarize and explain the proof to your group partner. See if you can make a drawing to help explain the approach.

2. Take turns explaining the proof you received to your partner. You get five minutes each.

3. What did you “get” about the proof your partner explained? What questions do you have?

22.3 Proof A

Let's say Y isn't connected —

so there are two open subsets $V, V' \subset Y$

with

- both non-empty
- $Y = V \cup V'$
- $V \cap V' = \emptyset$.

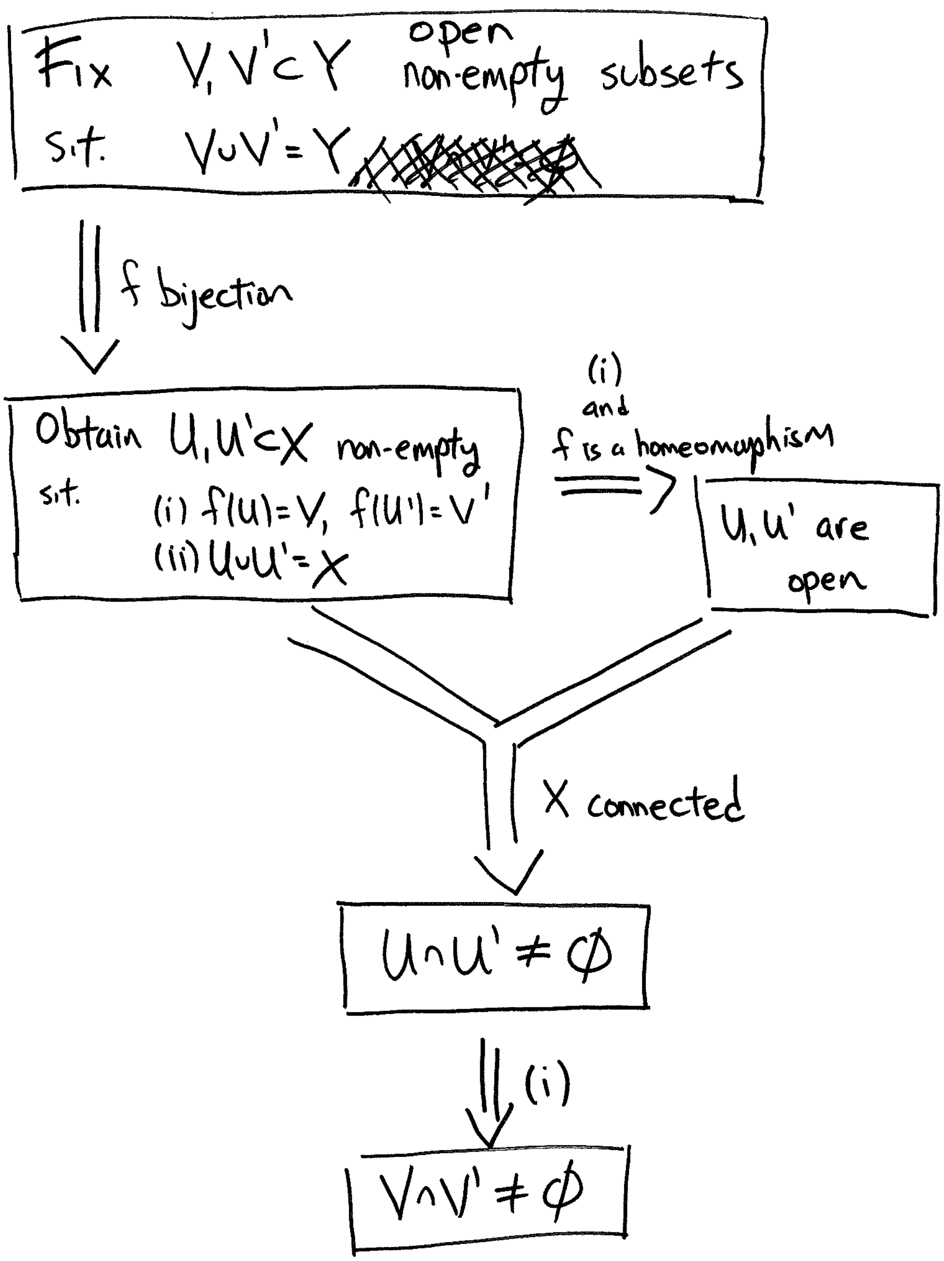
Then $U = f^{-1}(V)$ and $U' = f^{-1}(V')$

are open subsets of X satisfying

- U, U' are non-empty
- $X = U \cup U'$, and
- $U \cap U' = \emptyset$.

So X is not connected!

22.4 Proof B


Fix $V, V' \subset Y$ open non-empty subsets
s.t. $V \cup V' = Y$
f bijection
Obtain $U, U' \subset X$ non-empty
s.t. (i) $f(U) = V$, $f(U') = V'$
(ii) $U \cup U' = X$
(i) and f is a homeomorphism
U, U' are open
X connected
$U \cap U' \neq \emptyset$
(i)
$V \cap V' \neq \emptyset$

22.5 Comparing proofs

As a whole class:

1. What similarities do you see in the two proofs?
2. What differences do you see?
3. In Proof A, did we need all of the assumptions? How about in proof B?
4. Is it possible that the proposition could be made stronger? Changed in some way?

22.6 Testing modifications of statements

As a whole class:

1. Let's list some conjectures – other statements that might be true – based on what we notice about these proofs.
2. What would it mean to be able to *disprove* some of these conjectures?

22.7 Homework (producing proof)

For homework, you will choose again a conjecture, and try to prove it!