Reading 21

Connectedness

Today we're going to talk about connectedness.

21.1 Being open and closed in $[0, 1]$

For reasons that aren't obvious, let's see something interesting about the topology of $[0, 1]$:

Proposition 21.1.1. Suppose that $A \subset [0, 1]$ is a subset which is both closed and open. Then A is either empty, or equal to $[0, 1]$.

For this, we'll use a Lemma:

Lemma 21.1.2. If $B \subset [0, 1]$ is open, and if $b \in B$ does not equal 0 or 1, then there exists some $\epsilon > 0$ so that $(b - \epsilon, b + \epsilon) \subset B$.

Proof of Lemma 21.1.2. Since $B \subset [0, 1]$ is open, by definition of subspace topology, there exists $W \subset \mathbb{R}$ open so that $B = W \cap [0, 1]$. Now consider the intersection $W \cap (0, 1)$. This is an open subset of $\mathbb{R}$, being the intersection of two open subsets—in particular, for any $b \in W \cap (0, 1)$, there exists an open ball fully contained in $W \cap (0, 1)$ containing b. Let ϵ be the radius of this open ball. Then

$$(b - \epsilon, b + \epsilon) = \text{Ball}(b; \epsilon) \subset W \cap (0, 1) \subset W \cap [0, 1] = B.$$

□

Proof of Proposition 21.1.1. Suppose $B \subset [0,1]$ is closed and non-empty. Then B is in fact closed as a subset of $\mathbb{R}$.[1] B is obviously bounded, so it follows that B is compact by Heine-Borel.

Since the inclusion map $B \to \mathbb{R}$ is continuous, the extreme value theorem (Theorem 11.2.3) tells us that B has a maximal element, call it $b \in B$. If B is open, then b must equal 1, else Lemma 21.1.2 would contradict the maximality of b. Likewise, the minimal element in B must equal 0. Putting everythig together, We conclude that if B is open and closed and non-empty, then B contains 0 and contains 1.

This shows that B^C must *not* contain 0 and 1.

But now note that if B is open and closed, then B^C is open and closed. So if B^C is further non-empty, the previous paragraphs show that B^C contains 0 and 1.

We have shown that if B is open and closed, and if both B and B^C are non-empty, then B and B^C both contain 0 and 1. Of course, B and B^C cannot both contain 0 and 1. Thus, if B is open and closed, either B or B^C must be empty. □

This proposition is powerful. For example, we have the following:

Corollary 21.1.3. Let X be a discrete topological space and fix elements $x, x' \in X$. Then there exists a path from x to x' if and only if $x = x'$.

Proof. Suppose $\gamma : [0,1] \to X$ is continuous, and that x is in the image of γ. because X has the discrete topology, the singleton set $\{x\}$ is both closed and open. (To see this, recall that every subset of X is open in the discrete topology. In particular, both $\{x\}$ and its complement are open.) Thus, the preimage $\gamma^{-1}(\{x\})$ is both a closed and open subset of $[0,1]$. By Lemma 21.1.1, the preimage must be either empty or all of $[0,1]$. Because we assumed x to be in the image,

$$\gamma^{-1}(\{x\}) = [0,1].$$

In particular, γ is a constant function, so $\gamma(0) = \gamma(1) = x$. □

Example 21.1.4. So, if X is a discrete topological space with two or more elements, X is not path-connected.

[1] To see this, note that $B^C = W \cap [0,1]$ for some $W \subset \mathbb{R}$ open, by definition of subspace topology. Then we can check that $\mathbb{R} \setminus B = W \bigcup (\mathbb{R} \setminus B)$, so that B is open in $\mathbb{R}$. In fact, if $I \subset X$ is closed, then $B \subset I$ is closed if and only if B is also closed as a subset of X.

21.2 Connectedness

So, path-connectedness was an intuitive notion: We'll say a space is path-connected if any two points can be connected by a path. Confusingly, the term "path-connected" is not the same as the term "connected" in our culture.

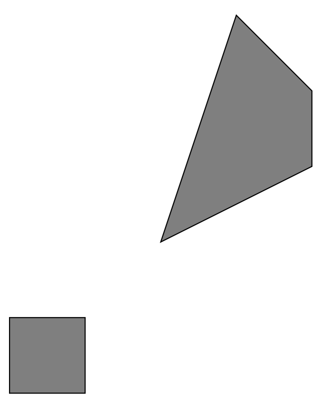
We now discuss a far less intuitive notion:

Definition 21.2.1 (Connected.)**.** We say that a space X is *connected* if the following holds: If $A \subset X$ is both open and closed, then either $A = X$ or $A = \emptyset$.

Example 21.2.2. By Proposition 21.1.1, we know that $X = [0, 1]$ is a connected space.

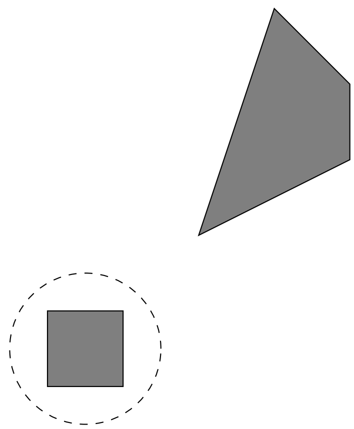
Example 21.2.3. Let X be a discrete topological space. If X has two or more elements, X is not connected.

Example 21.2.4. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

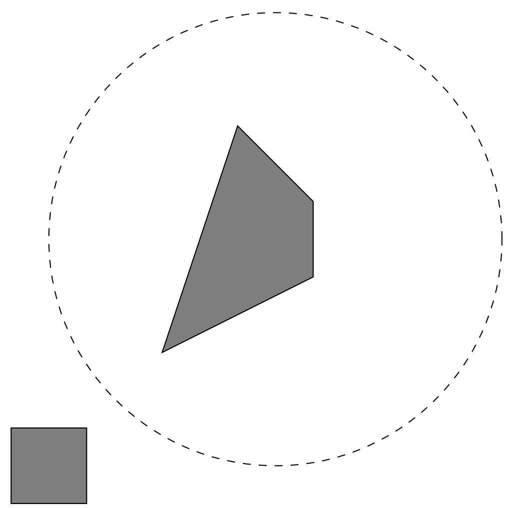


Let us label the lower-left component by A, and the upper-right component by B. I claim that both A and B are each both open and closed.

To see that A is open, simply observe that there is an open ball $W \subset \mathbb{R}^2$ for which $W \cap X = A$ (and then cite the definition of the subspace topology, which defines the topology on $X \subset RR^2$):



Because $B = A^C \subset X$, we conclude B is closed. To see B is open, likewise observe an open ball in X containing B but not A:



So B is open, meaning $A = B^C$ is closed. This shows $A \subset X$ is both open and closed, but $A \neq X$ and $A \neq \emptyset$.

Notice that all our examples connectedness/path-connectedness are the same. This is because of the following:

Proposition 21.2.5. If X is path-connected, then X is connected.

Proof. We will prove the contrapositive—that is, if X is not connected, then X is not path-connected.

Because X is not connected, there exists a subset $A \subset X$ which is non-empty, not all of X, but both open and closed.

So choose $x \in A$, and choose $x' \in A^C \subset X$. I claim there is no path from x to x'.

To see this, suppose we have a continuous map $\gamma : [0,1] \to X$ for which γ intersects A, we must have that $\gamma^{-1}(A)$ is non-empty. On the other hand, A is both open and closed, so $\gamma^{-1}(A)$ is both open and closed—this means $\gamma^{-1}(A) = [0,1]$ by Proposition 21.1.1.

That is, if $\gamma(t) \in A$ for some t, then $\gamma(t) \in A$ for every $t \in [0,1]$. In particular, if $x = \gamma(0)$, then $x' \neq \gamma(1)$. This proves the claim, and hence the proposition. □

Warning 21.2.6. There exist connected spaces that are not path-connected.

21.3 Being disconnected

Proposition 21.3.1. Let X be a topological space. The following are equivalent:

(a) X is not connected. (See Definition 21.2.1.)

(b) There exist two non-empty open subsets U, U' of X for which $U \cap U' = \emptyset$ and $X = U \cup U'$.

Definition 21.3.2 (Disconnected.)**.** If a topological space satisfies either of the equivalent conditions of Proposition 21.3.1, we say that X is *disconnected.*

21.4 Exercises

Exercise 21.4.1. Prove Proposition 21.3.1.