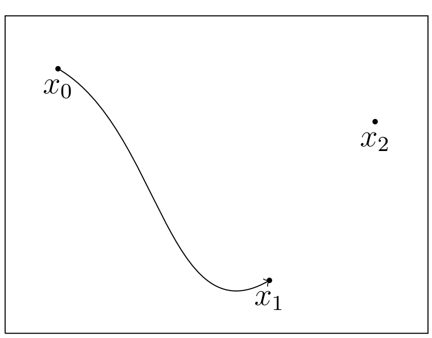
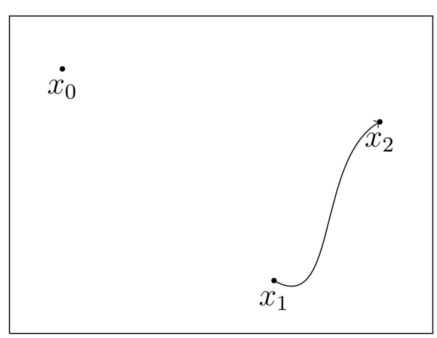
Reading 20

Invariance of domain and π_0

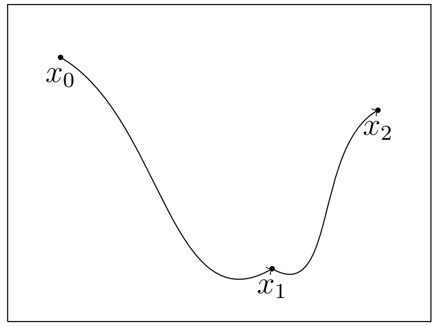
20.1 Concatenation of paths

Let X be a topological space, and choose three points x_0, x_1, x_2. Suppose you have a path from x_0 to x_1, and a path from x_1 to x_2.





Intuitively, it seems clear that there should be a path from x_0 to x_2. Indeed, why don't we just "do the first path, and then do the second?"



This is a great idea, but we need to turn the idea into a successful construction. Let's call this idea a "concatenation" of paths. What we need to do, then, is given two continuous paths that suitably agree at the endpoints, to construct a third called their concatenation. Here's how we'll define it.

Definition 20.1.1. Let $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$ be two continuous paths in X. Suppose that $\gamma(1) = \gamma'(0)$. Then the *concatenation* of γ with γ' is denoted

$$\gamma' \# \gamma$$

and is defined to be the function

$$t \mapsto \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \gamma'(2t-1) & t \in [1/2, 1]. \end{cases}$$

This is a bit much to parse, so let's talk it out. First, what does this function $\gamma' \# \gamma$ do when $t \leq 1/2$? It does exactly the path γ, but in double-time. (That is, twice as fast.) You might imagine a movie played on fast-forward, so that only the first half of the one-second allotted is used up to play the whole movie γ.

So what does the function do for $\geq 1/2$? The important thing to note here is that the function $t \mapsto 2t - 1$ is a bijection between $[1/2, 1]$ and $[0, 1]$. It sends $t = 1/2$ to 0, and sends $t = 1$ to 1. In other words, though $[1/2, 1]$ is an interval of only length $1/2$, the function $t \mapsto \gamma'(2t-1)$ "plays the entire movie of γ'" during the half-length time interval $[1/2, 1]$.

Informally, the concatenation $\gamma' \# \gamma$ "does γ at double-speed, then does γ' at double-speed." In particular, note that $\gamma' \# \gamma(0) = x_0$, and $\gamma' \# \gamma(1) = x_2$. And though we will not need this, note also that $\gamma' \# \gamma(1/2) = x_1$.

So now that we understand the concatenation as a function, we need to show that it is actually a path. That is, do we know that $\gamma'\#\gamma$ is continuous?

Proposition 20.1.2. Let $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$ be two continuous paths in X. Suppose that $\gamma(1) = \gamma'(0)$. Then $\gamma'\#\gamma$ is continuous.

Here is an informal reason as to why the proposition isn't obviously false: Because we know that $\gamma(1) = \gamma'(0)$, the concatenation $\gamma'\#\gamma$ doesn't have any "breaks" or "jumps." But this is informal. We'll need to actually prove that the concatenation is continuous using topology. (Note that X is an arbitrary topological space—it may not even be $\mathbb{R}^n$, and it could be some crazy metric space, or a crazy poset.)

To not lead ourselves astray, and to present the proof "the right way," we will delegate the proof to an add-on section at the end of this reading. You can just take Proposition 20.1.2 for granted.

Example 20.1.3. Let $m \geq 2$, and choose some point $x \in \mathbb{R}^m$. Let's prove that $X = \mathbb{R}^m \setminus \{x\}$ (that is, $\mathbb{R}^m$ with a point removed) is path-connected. (We are giving this set the subspace topology inherited from $\mathbb{R}^m$.)

Well, how did we prove $\mathbb{R}^m$ is path-connected? Given x' and x'', we defined a path by $(1-t)x' + tx''$. This is a fine, continuous path from x' to x'' in X so long as this line segment from x' to x'' never intersects x (i.e., so long as it doesn't pass through the point we removed). In other words, so long as x does not lie along the line interval from x' to x'', we have proven that there is a path in X from x' to x''.

Now suppose that x, x', x'' lie along a single line in $\mathbb{R}^n$ and that x is between x' and x'' along this line. Because $m \geq 2$, we can find some point y that does not lie on the line between x' and x''.[1] But then there is a path from x' to y, and a path from y to x'', that do not pass through y (e.g., straight line segments). By Proposition 20.1.2, the concatenation of these two paths produces a path from x' to x''. This finishes the proof.

20.2 Application: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$

Have you asked the question where $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$?

[1] This is the only place we are using the assumption on m.

Intuitively, we'd like to say that $\mathbb{R}$ could not be homeomorphic to $\mathbb{R}^2$; but we certainly haven't proven this in class. To give you some idea of the subtlety of this, note that there *do* exist bijections between $\mathbb{R}$ and $\mathbb{R}^2$.[2] In fact, there are many bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ (even when $m \neq n$).

In the earliest days of topology, we had the following basic question:

Question 20.2.2. If $m \neq n$, can $\mathbb{R}^m$ be homeomorphic to $\mathbb{R}^n$?

This is one of those basic questions that we think we ought to be able to answer. We can, now, but there were days when we couldn't! Now, we know that the question has the answer you expect: $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$. This fact (which is now a theorem) is often called *invariance of domain*.

Remark 20.2.3. Think of the days when we did not know the answer to the above question. Those days must have been both an exciting and anxious time. We were developing a tool called topology, but we didn't even know if it could answer the most basic questions!

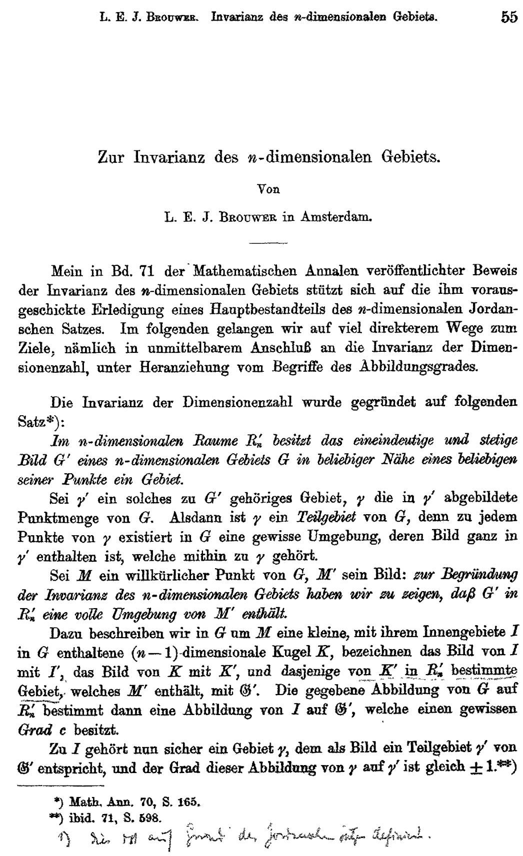
Indeed, what kinds of properties can distinguish $\mathbb{R}^n$ from $\mathbb{R}^m$ is a very good test – it did indeed turn out that the notion of "open set" allows for intuitions of dimension. If it didn't by now, it is possible we would have scrapped the whole idea of topological spaces and started anew.

The proof that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$ often goes through a tool called the Brouwer fixed point theorem, or a tool called homology, but we won't cover those topics just yet. (We will not cover homology in this course.)

Remark 20.2.4 (Some historical context)**.** The proof of invariance of domain is often credited to Brouwer, who published the proof in 1912 – see Figure 20.2.1. In fact, various attempts to prove invariance of domain had been made since at least 1878, if not earlier, pioneered by mathematicians such as Jacob Lüroth, Georg Cantor (you have heard of him!), and Eugen Netto.

To give some historical context: Maxwell published his equations on electromagnetism in 1873, the Michelson-Morley experiment was in 1878 (this

[2]This is a subtle fact that they might not always teach you in Math 3330. Here is one argument—$\mathbb{R}$ is in bijection with the set of subsets of $\mathbb{Z}_{>0}$. This shows that $\mathbb{R}^2$ is in bijection with the set of subsets of $\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0}$. But $\mathbb{Z}_{>0}$ is in bijection with $\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0}$, so $\mathcal{P}(\mathbb{Z}_{>0}) \cong \mathcal{P}(\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0})$.



L. E. J. Brouwer. Invarianz des n-dimensionalen Gebiets. 55

Zur Invarianz des n-dimensionalen Gebiets.

Von

L. E. J. Brouwer in Amsterdam.

Mein in Bd. 71 der Mathematischen Annalen veröffentlichter Beweis der Invarianz des n-dimensionalen Gebiets stützt sich auf die ihm vorausgeschickte Erledigung eines Hauptbestandteils des n-dimensionalen Jordanschen Satzes. Im folgenden gelangen wir auf viel direkterem Wege zum Ziele, nämlich in unmittelbarem Anschluß an die Invarianz der Dimensionenzahl, unter Heranziehung vom Begriffe des Abbildungsgrades.

Die Invarianz der Dimensionenzahl wurde gegründet auf folgenden Satz*):

Im n-dimensionalen Raume R_n' besitzt das eineindeutige und stetige Bild G' eines n-dimensionalen Gebiets G in beliebiger Nähe eines beliebigen seiner Punkte ein Gebiet.

Sei γ' ein solches zu G' gehöriges Gebiet, γ die in γ' abgebildete Punktmenge von G. Alsdann ist γ ein *Teilgebiet* von G, denn zu jedem Punkte von γ existiert in G eine gewisse Umgebung, deren Bild ganz in γ' enthalten ist, welche mithin zu γ gehört.

Sei M ein willkürlicher Punkt von G, M' sein Bild: *zur Begründung der Invarianz des n-dimensionalen Gebiets haben wir zu zeigen, daß G' in R_n' eine volle Umgebung von M' enthält.*

Dazu beschreiben wir in G um M eine kleine, mit ihrem Innengebiete I in G enthaltene $(n-1)$-dimensionale Kugel K, bezeichnen das Bild von I mit I',[1] das Bild von K mit K', und dasjenige von K' in R_n' bestimmte Gebiet, welches M' enthält, mit $\mathfrak{G}'$. Die gegebene Abbildung von G auf R_n' bestimmt dann eine Abbildung von I auf $\mathfrak{G}'$, welche einen gewissen *Grad* c besitzt.

Zu I gehört nun sicher ein Gebiet γ, dem als Bild ein Teilgebiet γ' von $\mathfrak{G}'$ entspricht, und der Grad dieser Abbildung von γ auf γ' ist gleich ± 1.**)

*) Math. Ann. 70, S. 165.
**) ibid. 71, S. 598.
1) Dies ist auf Grund des Jordanschen Satzes definiert.

Figure 20.2.1. An excerpt from Brouwer's paper proving Invariance of Domain, Mathematische Annalen volume 72, pages 55–56 (1912).

disproved the ether theory of light), X-rays were discovered by Röntgen in 1895, the electron was discovered in 1897, Einstein published special relativity in 1905, and Rutherford discovered the idea that atoms had nuclei in 1911.

So, the discovery that the language of topology sees the notion of dimension (invariance of domain) was made in the same era as some of the most fundamental discoveries of modern physics.

We prove a baby version of invariance of domain. It's an application of path-connectedness.

Theorem 20.2.5. $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}$ if and only if $m = 1$.

Proof of Theorem 20.2.5. Note that if $m = 0$, $\mathbb{R}^0$ is a point, so it's not even in bijection with $\mathbb{R}$.

Now suppose $m \geq 2$, and suppose $f : \mathbb{R}^m \to \mathbb{R}$ is a homeomorphism for the sake of contradiction. Choose a point $x \in \mathbb{R}^m$, and consider the space $U = \mathbb{R}^m \setminus \{x\}$ (given the subspace topology). Because f is a homeomorphism, we see that U is homeomorphic to $\mathbb{R} \setminus f(x)$.[3]

Note that $\mathbb{R} \setminus f(x)$ is not path connected. (This is a consequence of the intermediate value theorem.) Then Exercise 19.3.1 implies that U is not path-connected (because two homeomorphic spaces are either both path connected, or neither is path-connected). We have run into a contradiction, because Example 20.1.3 tells us that $\mathbb{R}^m \setminus \{x\}$ is path-connected. $\square$

Remark 20.2.6. Could we apply the arguments of Theorem 20.2.5 to prove invariance of domain for $\mathbb{R}^n$ when $n > 1$?

You can see that this proof method fails when $n \geq 2$, because then $\mathbb{R}^n \setminus \{x\}$ is still path-connected.

One of the big victories of algebraic topology was the discovery of notions of "path-connectedness" the go beyond considering only paths, but also disks of higher dimensions. You might learn about these "higher homotopy groups" if you take an advanced algebraic topology class.

[3]Here is the argument. The composition $U \to \mathbb{R}^m \to \mathbb{R}$ is continuous because composition of continuous functions is continuous. The composition has image $f(U)$, so by the universal property of the subspace topology for $f(U) \subset \mathbb{R}$, we obtain a continuous map $U \to f(U)$. Running the same argument for the inverse to f (to see that the inverse is a continuous map $f(U) \to U$) proves that $f(U)$ and U are homeomorphic.

20.3 Closed intervals and time reversal

By definition, the notion of path-connectedness depends on the topology of $[0, 1]$ (because we need to know which functions out of $[0, 1]$ are continuous). So let's study how we can think about $[0, 1]$, and other intervals, as a topological space, along with some cool things we can do with intervals.

We'll think about the interval $[0, 1]$ as parametrizing time. For example, we'll think of a continuous function

$$\gamma : [0, 1] \to X, \qquad t \mapsto \gamma(t)$$

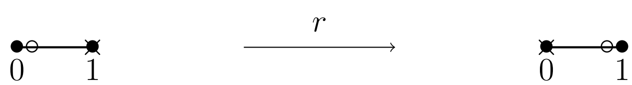
as giving a point $\gamma(t)$ inside of X for every "time" $t \in [0, 1]$.

Proposition 20.3.1 (Time reversal is continuous)**.** Consider the function

$$r : [0, 1] \to [0, 1], \qquad r(t) = 1 - t.$$

1. r is continuous.
2. In fact, r is a homeomorphism.

Remark 20.3.2. The above proposition tells us that "reversing time" is a continuous operation, and that it can be undone. Here is a picture of r:



The markings (the open dot, the closed dot, and the closed square) indicate which points of the domain are sent to which points in the codomain.

Proof. (1) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = 1 - t$ is continuous. (An example proof: f is a polynomial, and polynomial functions are continuous. Another proof can be obtained by using ϵ-δ; any $\delta > 0$ with $\delta \leq \epsilon$ will do.)

By definition of subspace topology, the inclusion $i : [0, 1] \to \mathbb{R}$ is continuous, so the composition $f \circ i$ is continuous.

You can check that $f \circ i$ has image given by $B = [0, 1]$. Hence, by the universal property for the subspace topology of B, we have a continuous function as in the dashed arrow:

$$\begin{tikzcd}
{[0,1]} \arrow[dr, dashed, "f'"'] \arrow[drr, "f \circ i"] & & \\
& B = {[0,1]} \arrow[r, "i_B"'] & \mathbb{R}
\end{tikzcd}$$

(Of course, B is equal to $[0,1]$, but I used the notation B to make clear *how* I was using the universal property.) This dashed arrow satisfies the property that $i_B \circ f' = f \circ i$. In other words,

$$f'(t) = i_B(f'(t)) = f(i(t)) = f(t) = 1 - t.$$

That is, $f' = r$. This shows r is continuous.

(2) Now, we can write down the inverse function to r straightforwardly (using algebra). For example, if a function s is an inverse to r, then

$$t = r(s(t)) = 1 - s(t)$$

so

$$s(t) = 1 - t.$$

(What we see is that r is its own inverse!) In particular, $s(r(t))$ also equals t, so s is both a right and left inverse to r. This proves that r is a bijection.

Finally, we know that r^{-1} is given by r, so from (1), we conclude that the inverse function to r is continuous. This shows that r is a homeomorphism. $\square$

Remark 20.3.3. The method of proof for (1) extends more generally: If $f : \mathbb{R} \to \mathbb{R}$ is any continuous function, and if $A \subset \mathbb{R}$ is any subset, then the composition $f \circ i_A$ is continuous, and if $B = f(A)$, then the induced function from A to B is continuous (giving both A and B the subspace topology).

Example 20.3.4. Here is an application of the proposition. Let $\gamma : [0,1] \to X$ be a continuous function to some topological space X. We can think of this as a "continuous movie" of the point $\gamma(0)$ traveling to the point $\gamma(1)$.

Then $\gamma \circ r$ is also a continuous function to X (because r is continuous, and compositions of continuous functions are continuous).

In other words, if we can depict a "continuous movie" of $\gamma(0)$ traveling to $\gamma(1)$, then we can depict a continuous movie of the reverse; $\gamma(1)$ traveling to $\gamma(1)$ (along the "same path," but backward).

Proposition 20.3.5. Let X be a topological space, and fix $x, x' \in X$. If there exists a path from x to x', then there exists a path from x' to x.

This should be an intuitive proposition: If there's a path from x to x', you can just "reverse" the path to get from x' to x. That's the intuition we'll follow in the proof.

Proof. Let

$$\gamma : [0,1] \to X$$

be a path from x to x' (so $\gamma(0) = x$, and $\gamma(1) = x'$). Let us define

$$\overline{\gamma} = \gamma \circ r.$$

Because r and γ are continuous, the composition $\overline{\gamma}$ is. Moreover,

$$\overline{\gamma}(0) = \gamma(f(0)) = \gamma(1) = x'$$

and likewise, $\overline{\gamma}(1) = x$. Thus $\overline{\gamma}$ is a path from x' to x. □

Remark 20.3.6. Let X be a topological space. Then for any $x \in X$, there exists a path from x to itself. To see this, note that the constant path

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x \forall t \in [0,1]$$

is a path from x to itself.

20.4 Path-connected components

Let's collect our knowledge about paths between points so far.

(i) Let $x \in X$. Then there is a path from x to itself. (For example, the constant path.)

(ii) Let $x, x' \in X$. If there is a path from x to x', then there is a path from x' to x. (For example, by time reversal. See Proposition 20.3.5.)

(iii) Let $x, x', x'' \in X$. If there is a path from x to x', and if there is a path from x' to x'', then there is a path from x to x''. (For example, by concatenation. See Proposition 20.1.2.)

All this is to say that there is an equivalence relation on any topological space X given as follows:

Theorem 20.4.1. Let X be a topological space, and consider the relation

$$x \sim x' \text{ if and only if there exists a path from } x \text{ to } x'.$$

Then $\sim$ is an equivalence relation.

There is a name for the set of equivalence classes for this relation:

Definition 20.4.2. Consider the equivalence relation from Theorem 20.4.1. We let

$$\pi_0(X) = X/\sim$$

denote the set of equivalence classes of $\sim$. We call an element of $\pi_0(X)$ a *path-connected component* of X, and we call $\pi_0(X)$ the set of path-connected components of X.

Remark 20.4.3. $\pi_0(X)$ is read "pie nought of X."

Proposition 20.4.4. Let $f : X \to Y$ be a continuous map. Then

(a) The assignment $\pi_0(X) \to \pi_0(Y)$ sending $[x] \mapsto [f(x)]$ is well-defined. In other words, f induces a function $\pi_0(X) \to \pi_0(Y)$.

(b) Moreover, if f is a homeomorphism, the induced function $\pi_0(X) \to \pi_0(Y)$ is a bijection.

Using this language, we can prove Theorem 20.2.5 more succinctly:

Proof of Theorem 20.2.5 using π_0 notation. For any $x \in \mathbb{R}$, we know that $\pi_0(\mathbb{R} \setminus \{x\})$ has more than one element. On the other hand, for all $m \geq 2$ and for any $y \in \mathbb{R}^m$, we know that $\pi_0(\mathbb{R}^m \setminus \{y\}$ has exactly one element. Because a homeomorphism $f : \mathbb{R} \to \mathbb{R}^m$ induces a homeomorphism $\mathbb{R}\setminus\{x\} \to \mathbb{R} \setminus \{f(x)\}$, and because a homeomorphism induces a bijection on π_0, we are finished. $\square$

20.5 Exercises

Exercise 20.5.1. Here, I ask you to prove a souped up version of Proposition 20.4.4.

Let $f : X \to Y$ be a continuous function.

(a) Show that $[x] \mapsto [f(x)]$ is a well-defined function from $\pi_0(X)$ to $\pi_0(Y)$.

We will call this function $f_\sharp$.

(b) Show that if $g : Y \to Z$ is another continuous function, then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$.

(c) Let $\mathrm{id}_X : X \to X$ be the identity function. Show that $(\mathrm{id}_X)_\sharp = \mathrm{id}_{\pi_0(X)}$.

(d) Show that if $f : X \to Y$ is a homeomorphism, then $f_\sharp$ is a bijection.

Add-on: Concatenating intervals, coproducts, and paths

Consider the interval $[0, 2]$. You may not have noticed this before, but you can think of $K = [0, 2]$ as "glued" out of the intervals $I = [0, 1]$ and $J = [1, 2]$ simply by identifying the element $1 \in I$ with the element $1 \in J$. The process of taking two intervals I and J, and gluing the right-endpoint of I to the left-endpoint of J to produce a new interval K, is called *concatenation*.

Now, the following caveat is both powerful and confusing. Even if I and J are the same interval, we can still consider what happens when we *treat* I and J as separate intervals, and glue I to J along the appropriate endpoints.

Example 20.5.2. Let $I = [0, 3]$ and $J = [0, 3]$. We can concatenate I and J to obtain an interval equivalent to the interval $[0, 6]$. More generally, if $I = J = [0, t]$, then the concatenation of I and J results in an interval equivalent to $[0, 2t]$.

This is secretly very important for us, but let me relegate its explanation to an add-on section for those of you who are curious–Section 20.6. I think the idea of concatenation is intuitive enough that we can move forward without needing to speak of coproducts.

20.6 Coproducts

The formal process of taking two sets, and ignoring whatever overlap they may have, is called taking the *disjoint union*, or taking the *coproduct*, of the two sets. We denote the disjoint union of I and J by

$$I \coprod J.$$

That symbol is literally an upside-down capital Pi.[4] The symbol $\coprod$ is also sometimes called the *coproduct* symbol. (It is also an upside down product symbol.[5])

We won't go too in-depth about coproducts in this rendition of the course, though you can look at course notes from a previous rendition of this course if you want more details.

The point of coproducts is to do something we can't easily do in physical reality. Suppose A and B are two sets, and that they happen to have some overlap (so $A \cap B \neq \emptyset$).

For our discussion, you might imagine that A is some region of a map, while B is another region, and the two regions overlap.

If you were to physically "cut out" A from your map, you'd of course take along some portion of B into the cut-out. ($A \cap B$ would be a subset of your cut-out.) In other words, you can't physically separate A from B.

What the operation of coproduct does is it "clones" both A and B. Here's what a coproduct would be in our physical example. Imagine somebody taking a photocopy of region A, and then a photocopy of region B. Then these photocopies are a copy of A and a copy of B that no longer physically overlap! This *photocopied* collection is how you can think about the coproduct of A and B.

So $A \cup B$ is the *actual union* of the regions A and B on the map. But $A \coprod B$ is the *union of the photocopies.* As an example, if $A \cap B$ contained Kazakhstan, then $A \coprod B$ would have *two* photocopied copies of Kashakstan, while $A \cup B$ would have only one.

In particular, note that in general,

$$A \cup B \neq A \coprod B.$$

[4] A capital Pi looks like Π (a big, rigid π).

[5] Recall from a previous class that we denote products using capital Pi: Π, just as in algebra, we denote sums using capital Sigma: Σ.

For example, if $A \cup B$ is a set with 5 elements, and if $A \cap B$ has 2 elements, then $A \coprod B$ has 7 elements.

Here is a rigorous definition in case you are curious:

Definition 20.6.1. Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. The *coproduct*, or *disjoint union*, of this collection is the set

$$\{(x, \alpha) \mid x \in X_\alpha\} \subset \left(\bigcup_{\alpha \in \mathcal{A}} X_\alpha \right) \times \mathcal{A}.$$

The disjoint union is defined by

$$\coprod_{\alpha \in \mathcal{A}} X_\alpha.$$

Remark 20.6.2. The big set $(\bigcup_{\alpha \in \mathcal{A}} X_\alpha) \times \mathcal{A}$ can be thought of as taking the (usual) union of all the X_α, then taking $\mathcal{A}$-many photocopies of that union. Of course, you don't need photocopies of *entire* maps, but only of the specified regions X_α. So for every αth photocopy, the disjoint union only contains those x in the αth photocopy that are contained in X_α.

And, we can define a topology on the coproduct, called the *coproduct topology.*

20.6.1 Concatenating intervals, revisited

Definition 20.6.3. Let $I = [a, b]$ and $J = [a', b']$ be two closed intervals. Then the *concatenation* of I with J is the topological space

$$\left(I \coprod J \right) / \sim,$$

where $\sim$ is the equivalence relation given as follows:

$$s \sim t \iff \begin{cases} s = t & \text{or} \\ s = b \in I \& t = a \in J & \text{or} \\ t = b \in I \& s = a \in J & . \end{cases}$$

(In words, we are gluing the rightmost endpoint of I to the leftmost endpoint of J.) Note that we are treating I and J as non-overlapping photocopies when we take the coproduct.

Proposition 20.6.4. The concatenation of I with J is homeomorphic to an interval of length $(b - a) + (b' - a')$.

20.6.2 All closed intervals of positive length are homeomorphic

Intuitively, two closed intervals look identical; but they may have different lengths. Below may be a first hint about how topologies do *not* care about geometric measurements like length, but only care about notions like shape. The Proposition states that any two closed intervals are always homeomorphic (so long as they each have positive, finite length).

Proposition 20.6.5. Let $[a, b]$ and $[a', b']$ be two closed intervals, and assume that $a - b \neq 0, b' - a' \neq 0$. [6]

Then $[a, b]$ and $[a', b']$ are homeomorphic.

Proof. We have a linear function f sending a to a' and b to b'. [7] The formula for f is

$$f(t) = \frac{b' - a'}{b - a}(t - a) + a'.$$

(Note that $b - a \neq 0$ by assumption.) You can check that the interval $[a, b]$ has image $[a', b']$ under this function, so the same techniques as in the previous proof (see Remark 20.3.3) shows that f is a continuous function from $[a, b]$ to $[a', b']$.

On the other hand, we can produce an inverse to f. Let's call it g. The formula is

$$g(t) = \frac{b - a}{b' - a'}(t - a') + a.$$

(One way to reason out this formula quickly: g should be the linear function taking a' to a, and taking b' to b.) The same arguments as before show that g defines a continuous function from $[a', b']$ to $[a, b]$. □

[6] Here, $a, a', b, b' \in \mathbb{R}$. In particular, all intervals are finite-length. Note also that every interval has some *positive*—i.e., non-zero—length.

[7] The two points (a, a') and (b, b') determine a line in $\mathbb{R}^2$; this line is the graph of f.