# Reading 19

# Path-connectedness

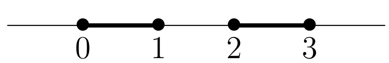
We are really building up some tools to study spaces. Today, we'll talk about the idea of what it means for a space to be *path-connected*. It's another nice property of a topological space. It will also lead to our first "invariant" of a topological space!

## 19.1 Path-connectedness

Let's say you're in a topological space $X$, and fix two points $x_0$ and $x_1$. Is it possible to "continuously walk" from $x_0$ to $x_1$? Let's first try to define what such a continuous walk would be.

We begin with an example.

**Example 19.1.1.** Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below:




Would you call $X$ connected?

**Remark 19.1.2** (Properties of spaces vs. properties of subsets)**.** Above, I used that $X$ was a subset of $\mathbb{R}$ to define the topology of $X$, but once we know about $X$'s topology, we could ask the connectedness question of $X$ (without reference to $\mathbb{R}$). Is the following space connected?

(Importantly, the picture makes no reference to $\mathbb{R}$ itself.) So unlike "closed" or "open," the adjective "connected" makes sense as a property of a space $X$. And when we ask whether a subset is connected, we are asking about the property of that subset as a space (endowed with the subspace topology). Aside from specifying the topology of the subspace, the parent set is irrelevant to the question of connectedness.

I want to talk today about two different ways to talk about the *connectedness* of a topological space.

First, some preliminaries: We let

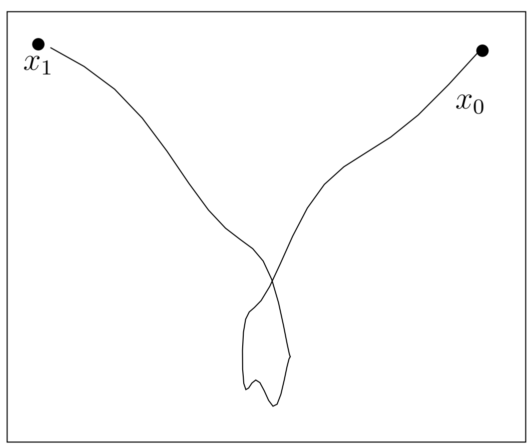
$$[0,1]$$

denote the usual closed interval from 0 to 1. We treat it as a topological space by giving it the subspace topology inherited from $\mathbb{R}$.

**Definition 19.1.3.** Let $X$ be a topological space. A *continuous path* (or *path* for short) in $X$ is a continuous function

$$\gamma : [0,1] \to X.$$

**Example 19.1.4.** Below is an image of a possible path $\gamma : [0,1] \to \mathbb{R}^2$.




Note that a path need not be injective (it can cross over itself).

**Definition 19.1.5.** Let $X$ be a topological space, and fix a path $\gamma : [0,1] \to X$. We say that $\gamma$ is a path *from* $\gamma(0)$ *to* $\gamma(1)$.

The following is the most intuitive definition of connectedness:

**Definition 19.1.6.** Let $X$ be a topological space. We say that $X$ is *path-connected* if for any two points $x, x' \in X$, there exists a path from $x$ to $x'$.

We can straightforwardly check that path-connectedness is indeed a property of topological spaces preserved by homeomorphisms (see Exercise 19.3.1).

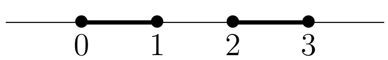
**Remark 19.1.7.** In fact, the proof method shows that if there is a continuous surjection from $X$ to $Y$, then the path-connectedness of $X$ implies the path-connectedness of $Y$.

## 19.2 Examples

$\mathbb{R}$ and $\mathbb{R}^n$ are path-connected. (See Exercises 19.3.2 and 19.3.3.)

Let's also see some examples of spaces that are *not* path-connected. In the following examples, the main tool we use will be the *intermediate value theorem* from calculus.

**Example 19.2.1.** Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below as before:




Then $X$ is not path-connected.

Indeed, I'll take $x$ to be some point in $[0,1]$ and $x'$ to be some point in $[2,3]$. Suppose (for the purpose of contradiction) that there is a path

$$\gamma : [0,1] \to X$$

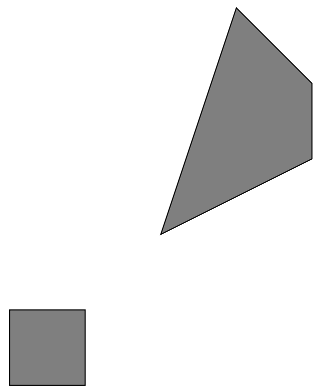
from $x$ to $x'$. Then the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}$$

(where the second map is the inclusion map) is continuous. By the intermediate value theorem from calculus, for any value $y$ such that $x \leq y \leq x'$, there must be some $t \in [0,1]$ such that $f(t) = y$.

But $\gamma$ has image contained in $X$, and in particular, the composition $f$ has no image in the open interval $(1,2)$. In particular, we have been led to a contradiction.

**Example 19.2.2.** Let $X$ be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

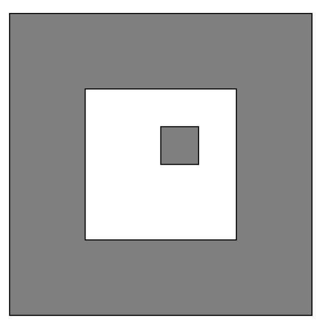


Then $X$ is not path-connected. The proof is similar to the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from $x$ to $x'$, where $x$ is in the lower-right component of $X$ and $x'$ is in the upper-left component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is the projection map sending $(x_1, x_2) \mapsto x_1$. Then $f$ is continuous, being a composition of continuous functions; but again, $f$ will violate the intermediate value theorem.

**Example 19.2.3.** Let $X$ be the subset of $\mathbb{R}^2$ shaded below, given the subspace topology:



Then $X$ is not path-connected. The proof is similar to the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from $x$ to $x'$, where $x$ is in the middle component of $X$ and

$x'$ is in the outer component. Then consider the composition

$$f : [0, 1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is now the map sending an element $y \in \mathbb{R}^2$ to the number $d(x, y)$. Then $f$ again violates the intermediate value theorem.

(Here, we are using the very nice fact that $d(x, -)$ is a continuous function on any metric space.)

**Remark 19.2.4.** In studying path-connectedness, we may draw pictures or use arguments reminiscent of analysis class. This is because of the central role of the real line in these discussions ($[0, 1]$ is a subspace of $\mathbb{R}$), and because your analysis class is devoted to the study of the real line.

## 19.3 Exercises

**Exercise 19.3.1.** Prove: If $X$ and $Y$ are homeomorphic, then $X$ is path-connected if and only if $Y$ is.

**Exercise 19.3.2.** Let $X = \mathbb{R}$. Show $X$ is path-connected.

**Exercise 19.3.3.** More generally, let $X = \mathbb{R}^n$ (with the standard topology). Show $X$ is path-connected.

**Exercise 19.3.4.** For any $n \geq 1$, let $X$ be the union of the origin and $S^n$; we give $X$ the subspace topology inherited from $\mathbb{R}^{n+1}$. Prove that $X$ is not path-connected.

*Possible solution for Exercise 19.3.1.* Suppose $X$ is path connected. We must show $Y$ is path-connected. So choose $y_0, y_1 \in Y$. We must exhibit a continuous path from $y_0$ to $y_1$. To do this, let $f : X \to Y$ be a homeomorphism, $g$ the inverse to $f$, and let $x_i = g(x_i)$. Because $X$ is path-connected, there is a continuous map $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = x_0$ and $\gamma_1 = x_1$. Because $f$ is continuous, $f \circ \gamma$ is a continuous function from $f(x_0)$ to $f(x_1)$. We are finished by observing that $f(x_i) = y_i$.

If $Y$ is path-connected, we can see that $X$ is path-connected by the same argument. $\square$

*Possible solution for Exercise 19.3.2.* To see this, fix any two points $x, x' \in X$. Then define a function $\gamma$ by "drawing a straight path from $x$ to $x'$." The previous sentence was vague, so let's make it precise: Define

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x + t(x' - x).$$

Note that $x$ and $x'$ are constants (we've fixed them!) while $t$ is the variable.

$\gamma$ is a continuous function. Let's shed some light on why: Because we've given $[0,1]$ the subspace topology, the inclusion

$$[0,1] \to \mathbb{R}, \qquad t \mapsto t$$

is a continuous function. Now let $f : \mathbb{R} \to \mathbb{R}$ be the linear function $t \mapsto x + t(x' - x)$. This is continuous (for example, it's a polynomial). Hence the composition

$$[0,1] \to \mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. On the other hand, this composition is precisely $\gamma$.

Finally, note that $\gamma(0) = x$ and $\gamma(1) = x'$. □

*Possible solution for Exercise 19.3.3.* To see this, given $x$ and $x'$ in $X$, again define

$$\gamma : [0,1] \to \mathbb{R}^n, \qquad t \mapsto x + t(x' - x).$$

Note now that we are using vector scaling and vector addition/subtraction to define $\gamma$. This is continuous because because the standard topology on $\mathbb{R}^n$ is the product topology. By the universal property of the product topology, to check the continuity of $\gamma$, we only need to check that every component of $\gamma$ is continuous. That is, we just need to check that the $n$ functions

$$t \mapsto x_1 + t(x_1' - x_1), \qquad \ldots, \qquad t \mapsto x_n + t(x_n' - x_n)$$

are continuous. But we saw this already in Exercise 19.3.2!

So $\gamma$ is continuous, and $\gamma$ is a path from $x$ to $x'$. □