Reading 11

Compactness, III. Extreme value theorem.

11.1 John the aspiring ballerino

Here's a story.

John loved ballet. He would go watch ballet twice a week, over 90 minutes each time. He loved the movements, he loved the music, he loved everything about ballet and watched with a passion. He longed to be a ballerino.

He signed up for an audition for the local ballet troupe. On the day of his audition, he got up on stage. He was asked if he was familiar with Swan Lake. He said yes, he'd seen it over and over. He had watched YouTube videos analyzing dancers, had heard dancers speak about their experiences performing Swan Lake, he had seen it all.

John was asked to perform any sequence from it.

And of course, he could not. He had never danced on his own. He had never taken lessons, or had a teacher; but worst, he had never spent the hours necessary to practice, to watch himself in a mirror, to dance.

Watching lectures will not teach you math. You have to try to dance yourself, and to learn how to improve what you see in the mirror. When you take an exam, you will not be assessed for the time you spent as an audience. You will be assessed for the time you spent alone practicing.

Remark 11.1.1. Practice does not mean merely doing something over and over. Practice – as any successful musician or performer will tell you – is about introspection and self-reflection. Are you really doing this movement correctly? Are you accurately reflecting the choreography with your body? It takes time and mental energy to engage in practice, and you must do it.

11.2 Three results today

Today we'll cover three results.

Proposition 11.2.1. Let $A \subset \mathbb{R}$ be compact.[1] Then there exists a maximal element in A.

More precisely, there exists $a \in A$ such that for all $a' \in A$, $a \geq a'$.[2]

Proof. By the Heine-Borel theorem, A is both closed and bounded.

Because A is bounded, there exists some $r > 0$ such that A is contained in the interval $(-r, r) \subset \mathbb{R}$. In particular, there is some real number b for which $a' \in A \implies a' \leq b$. Let us call such a number b an *upper bound* for A. (Note that there are infinitely many upper bounds for A.) By the least upper bound property of the real line[3], the set of upper bounds of A has a minimal element called b_0. In other words, b_0 is the smallest real number satisfying the upper bound property.

I claim that b_0 is an element of A. (This would prove the proposition.) To see this, for every integer $n \geq 1$, we simply choose an element $a_n \in A$ such that $\mathrm{dist}(a_n, b_0) \leq 1/n$. This is possible because b_0 is the *least* upper bound.

Then the sequence $\{a_n\}_{n\geq 1}$ converges to b_0. (Given any $\epsilon > 0$, choose N large enough so that $1/N < \epsilon$, and we see that $\mathrm{dist}(a_n, b_0) < \epsilon$ for all $n \geq N$.)

But A is closed, so $b_0 \in A$. □

Proposition 11.2.2. Let X be compact, and $f : X \to Y$ a continuous function. Then the image of f (given the subspace topology inherited from Y) is compact.

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[1] By the Heine-Borel theorem, this means A is a closed and bounded subset of $\mathbb{R}$. Also as usual, when we say that A is compact, we are really endowing A with the subspace topology inherited from the standard topology on $\mathbb{R}$. This is important, because a set may admit many different topologies.

[2] Here we are using the usual order on $\mathbb{R}$—whether two numbers are less than or equal to each other.

[3] This is a property of the real line we won't go over in this class; you'll see it, or have learned about it, in analysis.

Combining the above results gives us a result you are familiar with from calculus; but we have generalized to the case where the domain is any compact topological space.

Theorem 11.2.3 (Extreme Value Theorem)**.** Let X be compact, and let $f : X \to \mathbb{R}$ be a continuous function. Then f attains a maximal value.

That is, there exists some $x \in X$ such that, for every $x' \in X$, we have that $f(x) \geq f(x')$.

Remark 11.2.4. The condition that X be compact is necessary. For example, let $X = \mathbb{R}$ and let $f : X \to \mathbb{R}$ be the identity function, so $f(x) = x$. This attains no maximal value.

For the rest of today, you prove Proposition 11.2.2 and the Extreme Value Theorem. You will do this even if you've read a proof before – this is similar to a dancer practicing until they can nail down a move, or to any other athlete/musician/artist honing their craft.

Exercise 11.2.5. Prove Proposition 11.2.2.

Exercise 11.2.6. Prove Theorem 11.2.3.

11.3 Interlude: Why compactness?

The notion of compactness is wonky. It shouldn't seem natural the first time you see it, because you have no idea how to use the notion.

Let me just say that, often, the "nicest" spaces to work with happen to be compact. Though we don't know it yet, the easiest spaces to think about are often compact.

The following will be imprecise.

If somebody asks you think of a set that you feel comfortable with, many of you may choose a set out of famiiliarity (like $\mathbb{Z}$ or $\mathbb{R}$), or out of "smallness" (like a set with two elements). Indeed, the sets that seem least scary to you are probably the finite sets. Knowing that a set is finite gives you some emotional comfort.

Moreover, knowing that a set is finite will put some restrictions on how hard proofs can be about that set. For example, any function f from a finite set A to $\mathbb{R}$ must have a minimum and maximum; after all, f takes on only finitely many values, so you can just compare them all to each other. (This strategy breaks down horribly if A is not finite, of course.)

As it turns out, compact spaces are as comforting to topologists as finite sets are to you.

Let me give some sophisticated indication as to why. By nature of the definition of compactness, this indication will make heavy use of the notion of open covers.

What are open covers good for? It turns out that you can think of any space as "glued" out of an open cover. What do I mean?

A useful analogy to keep in mind is how a world atlas can tell you how to reconstruct (the surface of) the earth. You can imagine that each page of an atlas represents an open subset of the surface of the earth. (It's not quite true; you want to remove the "boundary" edges of each page of the atlas to get an actual open subset of the surface of the earth.) Now, you could tear out every page of your beautiful atlas, and you could papier-mache the pages together in a way that's faithful to the way the world is. For example, Page 23 of the atlas might contain a region around Texas, and so might Page 35. Then in your papier-mache, you should make sure that those regions match up propertly.

The result—of gluing together pages of the atlas in a way that respects overlapping regions—will be a wet, clumpy, but nevertheless a representation of the (surface of the) earth.

Well, let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a space X. You might think of each U_α as some page of an atlas for X.

Claim. You can construct (a space homeomorphic to) X by gluing together the U_α.

What does this claim actually mean? As a set, we know that $\bigcup U_\alpha = X$ by definition of cover. But can we topologize the lefthand side in such a way that this equality actually becomes a homeomorphism?

Moreover, in our atlas analogy, the U_α were pages of an atlas, completely separate.

Here is what compactness of X allows for: Even if somebody gives you an atlas of X with an infinite number of pages, you can guarantee that you only need finitely many of those pages of the atlas to create X. In particular, to study X, you need only study things about finitely many pages of the atlas.

That this is true for *any* complicated atlas of X is the power of compactness.

11.4 Interlude: Showing sets are closed or bounded

Compact spaces are nice—we saw from the extreme value theorem that compact spaces are generalizations of closed, bounded intervals. The Heine-Borel theorem tells us that we should figure out ways to determine whether certain subsets $A \subset \mathbb{R}^n$ are closed, and whether they are bounded.

Remark 11.4.1. Some of the most important theorems in math tells us when two different criteria are equivalent. For example, if a space arises as a subspace of Euclidean space, the Heine-Borel theorem tells us that the idea of *compactness* (which has to do with open covers) is equivalent to the idea of being closed and bounded in Euclidean space (which has to do with complements being open, and being at most some finite distance away from the origin). These are two very different kinds of criteria.

So if two criteria are equivalent, which do you use?

This depends very much on *how* the space in question is defined. Sometimes, A is defined in a way where it seems very difficult to verify that A is a closed subset of $\mathbb{R}^n$, for example. In which case, the "every open cover admits a finite subcover" criterion may be easier to verify. In life, usually, the *way* that A is defined tells you the way you should prove A. If A is made of plastic, it's usually the case that plastic-y methods are effective at dealing with A. Likewise, if A is defined using particular kinds of language, that kind of language will be most effective at dealing with A.

The major advances in mathematics arise when people discover that a language quite separate from the original definition of a thing happens to be effective at studying that thing. The Heine-Borel theorem is an example of how the language of open covers is incredibly useful at studying very concrete problems about subspaces of Euclidean space.

11.4.1 Strategies for recognizing/proving that something is closed

So let's talk about ways to identify when a subset A of $\mathbb{R}^n$ might be *closed*.

1. The complement of A is open.

2. A is the intersection of (possibly infinitely many) other sets known to be closed.

3. A is the union of *finitely many* sets known to be closed.

4. A is the pre-image of a closed set under a continuous function. (This is common when A is defined by an equation. Often times, the equation involves only continuous functions. For example, if A is the set of those points (x_1, x_2) satisfying the equation $x_2^3 + x_1 = 3$, then you know that A is the preimage of the set $\{3\}$ under the function $(x_1, x_2) \mapsto x_1 + x_2^3$ from $\mathbb{R}^2$ to $\mathbb{R}$. This function is continuous because it is polynomial, so the preimage of $\{3\}$—which is a closed subset of $\mathbb{R}$—is also closed.)

5. For every sequence in A which converges to some element b in $\mathbb{R}^n$, one can conclude that $b \in A$.

As you can see, there are many ways to prove that A is closed when A is a subset of $\mathbb{R}^n$. (In fact, every method except the last is a valid method when A is a subspace of *any* topological space X).

Example 11.4.2. Prove that the n-simplex

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, \text{Each } x_i \geq 0, \text{ and } \sum_{i=0}^{n} x_i = 1\} \subset \mathbb{R}^{n+1}$$

is a closed subset of $\mathbb{R}^{n+1}$.

For this, I would not recommend trying to show that the complement of Δ^n is open, though it is possible.

Notice that the set Δ^n is defined using inequalities and equations. Inequalities and equations are signs that a set is defined as the preimage of something, while the fact that there are multiple conditions usually means that the set is an intersection of other sets.

So first, let's note that a point of Δ^n has to satisfy the following conditions:

- For every $i = 0, \ldots, n$, x_i must be non-negative.

- The sum $x_0 + x_1 + \ldots + x_n$ must equal 1.

The last condition tells us that Δ^n must be contained in the preimage of $\{1\}$ under the function

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad (x_0, x_1, \ldots, x_n) \mapsto x_0 + x_1 + \ldots + x_n.$$

The first condition tells us that Δ^n must be in the preimage of $[0, \infty)$ under the functions

$$p_i : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad (x_0, \ldots, x_n) \mapsto x_i$$

for each i. (There are functions $p_0, p_1, \ldots, p_n$.) The above functions are all continuous, while we've seen in previous classes that $\{1\}$ and $[0, \infty)$ are both closed subsets of $\mathbb{R}$.

Finally, the fact that a point is in Δ^n if and only if it is in all of the preimages above shows that Δ^n is the intersection of these closed subsets. Explicitly,

$$\Delta^n = f^{-1}(\{1\}) \bigcap p_0^{-1}([0, \infty)) \bigcap p_1^{-1}([0, \infty)) \bigcap \ldots \bigcap p_n^{-1}([0, \infty)).$$

As mentioned above, each of these sets in the intersection is known to be closed. So the intersection itself is closed. This shows that Δ^n is closed.

Possible solution for Exercise 11.2.5. We must prove that $f(X)$ is compact. Instead of the function $f : X \to Y$, consider the function $f' : X \to f(X)$ which sends any $x \in X$ to $f(x) \in f(X)$. The function f' is continuous by the universal property of the subspace topology (for $f(X)$).

Let $\mathcal{V}$ be any open cover of $f(X)$. Then because f' is continuous, and by the construction of pullback open covers,

$$\mathcal{U} := \{U \subset X \,|\, U = (f')^{-1}(V) \text{ for some } V \in \mathcal{V}\}$$

is an open cover for X. Because X is compact, we may choose a finite subcover $\{U_1, \ldots, U_n\}$. For each $i = 1, \ldots, n$, choose $V \in \mathcal{V}$ to be an open subset for which $U_i = (f')^{-1}(V_i)$.

I now claim that the collection $\{V_1, \ldots, V_n\}$ is a subcover of $\mathcal{V}$. It suffices to show that $V_1 \bigcup \ldots \bigcup V_n = f(X)$. To see this, choose $y \in f(X)$. Then by definition of image, there is some $x \in X$ for which $f(x) = y$. Because $\{U_1, \ldots, U_n\}$ is an open cover, there is some i for which $x \in U_i$. In other words, there is some i for which $x \in (f')^{-1}(V_i)$, meaning $f'(x) \in V_i$. But $f'(x) = f(x) = y$, so we conclude $y \in V_i$. This shows that the union $V_1 \bigcup \ldots \bigcup V_n$ contains $f(X)$; because this union is a priori a subset of $f(X)$, we see that the union equals $f(X)$.

This completes the proof. □

Possible solution for Exercise 11.2.6. The image $f(X) \subset \mathbb{R}$ is compact by Proposition 11.2.2. Thus it has a maximal value by Proposition 11.2.1. $\square$