

# Reading 10

## Compactness, II. Heine-Borel theorem.

Last time we learned about covers, open covers, and subcovers. After seeing examples, we culminated with the definition of compactness.

We saw that finite spaces are compact, that posets with minimal elements are compact, and (without proof) that  $[0, 1] \subset \mathbb{R}$  is compact when equipped with the subspace topology.

We saw that  $\mathbb{R}^n$  is not compact.

Today's main result (that you are expected to know) is the Heine-Borel theorem.

### 10.1 The Heine-Borel Theorem

Let's keep building up our examples of compact spaces. There's a theorem called the Heine-Borel theorem that actually tells us *exactly* all the subsets of  $\mathbb{R}^n$  that are compact (with respect to the subspace topology).

To state the theorem succinctly, we'll want to learn another term.

**Definition 10.1.1.** Let  $A \subset \mathbb{R}^n$ . We say that  $A$  is *bounded* if there exists some positive real number  $r > 0$  such that

$$A \subset \text{Ball}(0, r).$$

In other words, there is some big number  $r$  so that *every* point of  $A$  is at most  $r$  units away from  $A$ .

Equivalently,  $A$  is called bounded if there is some  $r > 0$  such that, for every  $a \in A$ , we have that  $\text{dist}(0, a)$ —the distance from the origin to  $a$ —is less than  $r$ .

Now, it turns out we can completely describe which subsets of  $\mathbb{R}^n$  are compact:

**Theorem 10.1.2** (Heine-Borel theorem). Let  $A$  be a subset of  $\mathbb{R}^n$ .

Then  $A$  is compact if and only if  $A$  is both closed and bounded.

**Remark 10.1.3.** Note that being closed and bounded are properties of  $A$  as a *subset* of  $\mathbb{R}^n$ . When we endow this subset with the subspace topology inherited from  $\mathbb{R}^n$ , we may make sense of what it means for  $A$  to be compact. And that is what we mean.

So, put in a more wordy fashion, the Heine-Borel theorem says:  $A$ , with the subspace topology, is compact if and only if  $A$  is closed and bounded as a subset of  $\mathbb{R}^n$ .

We won't prove the Heine-Borel theorem today. But you may use it freely from now on.

To make good use of the Heine-Borel theorem, we'll want tools to decide when a subset of  $\mathbb{R}^n$  is closed, and when it is bounded.

The homework will have you determine whether certain subsets of  $\mathbb{R}^n$  are closed, are bounded, both, or neither.

Because it's usually straightforward to decide whether a subset of  $\mathbb{R}^n$  is bounded or not (you just need to determine whether elements in that subset can be arbitrarily far away from the origin) we'll focus on a study of closed subsets of  $\mathbb{R}^n$ .

## 10.2 Closed subsets

Throughout this section, we're going to assume we have placed the standard topology on  $\mathbb{R}^n$ .

**Remark 10.2.1** (Two skills with which you must be proficient). By their very definition, closed subsets often require us to demonstrate something about complements of sets. So most proofs involving closed subsets will involve some ingredient of complements; so you will want to be fluent with computing complements.

On the other hand, once you understand a complement, to prove it is open, you must learn how to construct open balls (by definition of open in the standard topology of  $\mathbb{R}^n$ ).

### 10.2.1 Some easy closed subsets of $\mathbb{R}^n$

**Exercise 10.2.2.** Let  $A = \{x \in \mathbb{R}^n \mid \text{dist}(0, x) \geq 5\}$ . That is, this is the set of points that are distance 5 or greater from the origin.

Prove  $A$  is closed.

**Exercise 10.2.3.** Let  $A = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$ . This is otherwise known as the  $x_2$ -axis. Prove that  $A$  is closed.

**Proposition 10.2.4.** The following are all closed subsets of  $\mathbb{R}$ :

1. A set consisting of a single point.
2. For any real number  $a$ , the intervals  $[a, \infty)$  and  $(-\infty, a]$ .

**Exercise 10.2.5.** Prove Proposition 10.2.4.

### 10.2.2 Ways to make new closed subsets

Now that we have some singleton collections of closed subsets (of  $\mathbb{R}$  and of  $\mathbb{R}^n$ ) let's see if we can make some more sophisticated ones.

As I mentioned before, because the definition of closedness involves complements, so will many proofs. The above properties are a straightforward consequence of *de Morgan's laws*, which are the most useful tools for understanding unions and intersections of complements. We will assume these without proof:

**Proposition 10.2.6** (de Morgan's laws). The complement of a union is the intersection of the complements. And the union of complements is the complement of the intersection.

In symbols, for any collection of sets  $\{B_\alpha\}_{\alpha \in \mathcal{A}}$ , we have that

$$\left( \bigcup_{\alpha \in \mathcal{A}} B_\alpha \right)^C = \bigcap_{\alpha \in \mathcal{A}} B_\alpha^C, \quad \bigcup_{\alpha \in \mathcal{A}} B_\alpha^C = \left( \bigcap_{\alpha \in \mathcal{A}} B_\alpha \right)^C$$

(ii) and (iii) below helps us make new closed subsets out of old ones:

**Proposition 10.2.7.** Let  $X$  be a topological space with topology  $\mathcal{T}$ . Let  $\mathcal{K}$  be the collection of closed subsets of  $X$  (so that  $B \in \mathcal{K} \iff B^C \in \mathcal{T}$ ). Then the following hold:

- (i)  $\emptyset, X \in \mathcal{K}$ .
- (ii)  $\mathcal{K}$  is closed under finite unions. That is, if  $B_1, \dots, B_n$  are closed subsets of  $X$ , then the union  $B_1 \cup B_2 \dots \cup B_n$  is also a closed subset of  $X$ .
- (iii)  $\mathcal{K}$  is closed under arbitrary intersections. That is, if we have an arbitrary collection  $\{B_\alpha\}_{\alpha \in \mathcal{A}}$  of closed subsets of  $X$ , then the intersection  $\bigcap_{\alpha \in \mathcal{A}} B_\alpha$  is also a closed subset of  $X$ .

(ii), in plain English, says the finite union of closed sets is closed. (iii) says that the intersection of closed sets is closed. You see the similarity with the axioms of a topology, except that the finiteness condition is swapped.

**Exercise 10.2.8.** Prove Proposition 10.2.7.

**Exercise 10.2.9.** We saw earlier that  $[a, \infty) \subset \mathbb{R}$  is closed. Likewise, we know that  $(-\infty, b]$  is closed.

Assuming  $a > b$ , prove that the closed interval  $[a, b] \subset \mathbb{R}$  is closed (Hint: intersections of closed sets).

Given some real numbers  $a_1 < a_2 < \dots < a_{2n}$ , show that the union

$$[a_1, a_2] \cup [a_3, a_4] \cup \dots \cup [a_{2n-1}, a_{2n}]$$

is closed. (Hint: A finite union of closed subsets is...?)

Likewise, if  $A \subset \mathbb{R}$  is a finite subset, prove that  $A$  is closed. (Hint: A finite union of closed subsets is...?)

### 10.2.3 Making closed sets using preimages

The following is another method of creating closed subsets:

**Proposition 10.2.10.** Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow Y$  be a continuous function. Then if  $B \subset Y$  is closed, then  $f^{-1}(B)$  is closed.

**Exercise 10.2.11.** Prove Proposition 10.2.10.

**Remark 10.2.12.** In fact, a function  $f : X \rightarrow Y$  is continuous if and only if the preimage of any closed subset of  $Y$  is closed (Proposition 7.7.2). So this is an equivalent definition of continuity.

In other words, once you start having a wealth of closed subsets of some space  $Y$ , and a wealth of continuous functions to  $Y$ , then we can start discovering many closed subsets of  $X$ .

Given our wealth of continuous functions (see Example 7.6.4), we can now exhibit all kinds of closed subsets of  $\mathbb{R}^n$ .

**Exercise 10.2.13.** (a) Show  $S^{n-1} \subset \mathbb{R}^n$  is closed.

(Hint:  $S^{n-1}$  is the preimage of the one-element set  $\{1\} \subset \mathbb{R}$  (which is closed) with respect to the function  $x \mapsto \text{dist}(0, x)$  (which is continuous).)

(Bonus: Note that this sphere is also bounded, so  $S^{n-1}$  is compact when given the subspace topology inherited from  $\mathbb{R}^n$ .)

(b) Show the unit disk  $D^n \subset \mathbb{R}^n$  is closed.

(Hint:  $D^n$  is the preimage of the interval  $[0, 1]$  (which is closed in  $\mathbb{R}$ ) with respect to the function  $x \mapsto \text{dist}(0, x)$  (which is continuous).) (Bonus: Since  $D^n$  is bounded,  $D^n$  is compact by the Heine-Borel theorem.)

(c) Fix an integer  $i$  between 1 and  $n$ . Show that the “ $i$ th upper half space”  $\{x \in \mathbb{R}^n \mid x_i \geq 0\}$  is closed.

(Hint: This is the preimage of the interval  $[0, \infty)$  (which is closed in  $\mathbb{R}$ ) under the projection map  $x \mapsto x_i$  (which is continuous).) (Comment: This is not bounded, so the upper half space is closed and not compact.)

(d) Show the “non-negative octant”

$$\{x \in \mathbb{R}^n \mid \text{For all } i, x_i \geq 0\}$$

is closed.

(Hint: The octant is the intersection of the upper half-spaces from the previous example (and an intersection of closed sets is closed.) (Remark: This set is also not bounded.)

(e) Show that the hyperplane

$$\{(x_0, \dots, x_n) \mid \sum_{i=0}^n x_i = 1\} \subset \mathbb{R}^{n+1}$$

is closed.

(Hint: This set is the preimage of the singleton set  $\{1\}$  (which is closed in  $\mathbb{R}$ ) under the function  $(x_0, \dots, x_n) \mapsto x_0 + x_1 + \dots + x_n$  (which is continuous).) (Note: This set is not bounded.)

(f) Show that the  $n$ -simplex  $\Delta^n \subset \mathbb{R}^{n+1}$  is closed.

(Hint: The  $n$ -simplex can be expressed as the intersection of the hyperplane from the previous example with the positive octant.) (Bonus: This set is bounded, so the  $n$ -simplex (given the subspace topology) is compact by the Heine-Borel theorem.)

(g) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Show that the graph of  $f$ , which is defined to be the subset of  $\mathbb{R}^2$  defined by

$$\{(x_1, x_2) \mid f(x_1) - x_2 = 0\}$$

is a closed subset of  $\mathbb{R}^2$ .

(Hint: This is the preimage of  $\{0\}$  (which is a closed subset of  $\mathbb{R}$ ) under the function  $\mathbb{R}^2 \rightarrow \mathbb{R}, (x_1, x_2) \mapsto f(x_1) - x_2$ . This is a continuous map because it is obtained by combining continuous functions together. (The notion of “combining” is left vague for the moment.) (Note: This set is not bounded—for example, there are points on the graph of  $f$  with arbitrarily large  $x_1$  coordinate.)

(h) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function, fix a real number  $a$ , and consider the set

$$\{x \mid f(x) \geq a\}.$$

Show this is a closed subset of  $\mathbb{R}^n$ .

(Hint: The set is a pre-image of the set  $[a, \infty)$  (which is closed in  $\mathbb{R}$ ) under the map  $x \mapsto f(x) - a$ .)

You can take finite unions and arbitrary intersections of the above examples to produce many, many closed subsets of  $\mathbb{R}^n$ .

### 10.3    **A useful fact: You can pull back open covers**

**Proposition 10.3.1.** Let  $f : X \rightarrow Y$  be a continuous function, and let  $\mathcal{V}$  be an open cover of  $Y$ . Define

$$\mathcal{U} := \{U \subset X \mid U = f^{-1}(V) \text{ for some } V \in \mathcal{V}\}.$$

Then  $\mathcal{U}$  is an open cover of  $X$ .

The math lingo for this proposition is that open covers “pull back.” We call  $\mathcal{U}$  the *pull-back* of  $\mathcal{V}$ .

**Exercise 10.3.2.** Prove Proposition 10.3.1.

A special example of this is when  $A$  is a subset of  $X$ , and  $i_A : A \rightarrow X$  is the inclusion function. Then any open cover of the big space  $X$  “pulls back” to an open cover of  $A$ . In this case,  $i_A^{-1}(V)$  has a particular simple interpretation, as

$$i_A^{-1}(V) = A \cap V.$$

## 10.4 Useful facts about compactness

We haven’t seen yet why compactness is useful. Here is a first sign:

**Proposition 10.4.1.** Let  $X$  be compact.<sup>1</sup> Fix a continuous function  $f : X \rightarrow Y$ . Then the image of  $f$ —endowed with the subspace topology inherited from  $Y$ —is compact.

In words, the image of a compact space is compact (so long as the image is taken with respect to a continuous function).

**Exercise 10.4.2.** Prove Proposition 10.4.1.

**Remark 10.4.3.** Thus, compactness is one of the few properties that “push forward” under a continuous map. Usually, preimages behave well under continuity by definition. We’ve already seen examples where preimages of compact spaces need not be compact, but here we see that images of compact spaces are always compact.

In the above proof, we have put together some of the wonderful ingredients we’ve learned so far—the universal property of subspaces, and that open covers pull back, for example.

Here is a corollary of the above fact:

**Corollary 10.4.4.** Let  $X$  be a compact topological space, and let  $f : X \rightarrow \mathbb{R}$  be a continuous function. Then  $f(X)$  is compact.

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<sup>1</sup>Note that at this point, we may infer that  $X$  is a topological space; this is because “compact” is an adjective that only makes sense for topological spaces.

Now, let me state that it is *incredibly common* to study functions from a space into  $\mathbb{R}$ . For example, if  $X$  is the set consisting of points on the surface of the earth,  $f : X \rightarrow \mathbb{R}$  could be a function sending a point  $x$  to the height of  $x$  above/below sea level. Or it could send  $x$  to the temperature at  $x$ , or the pressure, et cetera.

As you well know, there are points of “highest elevation” on earth. It also turns out that, if you assume the temperature function is continuous, there will (at any given moment time) be a point of highest and lowest temperature. This is a consequence of the fact that we can say what all compact subsets of  $\mathbb{R}$  must look like! It turns out that every compact subset of  $\mathbb{R}$  contains a maximal and minimal element. We will get to that later.

Here is another useful fact.

**Proposition 10.4.5.** Let  $X$  be a compact topological space, and let  $A \subset X$  be a closed subset. Then  $A$  (with the subspace topology inherited from  $X$ ) is compact.

In other words, closed subsets of compact spaces are compact.

*Proof.* We must show that every open cover of  $A$  admits a finite subcover.

Choose an open cover  $\mathcal{U}$  of  $A$ . By definition of subspace topology, for every  $U \in \mathcal{U}$  there is some  $V \in \mathcal{T}_X$  such that  $U = V \cap A$ .<sup>2</sup> Choose one  $V$  for every  $U \in \mathcal{U}$ , and let  $\mathcal{V}$  denote the collection of these chosen  $V$ . (So  $\mathcal{V}$  is in bijection with  $\mathcal{U}$ .)

Now consider the collection  $\mathcal{W} = \mathcal{V} \cup \{A^C\}$ . That is,  $\mathcal{W}$  consists of the sets  $V$ , and of another set called  $A^C$ . Note that  $A^C$  is open because  $A$  is closed; so every member<sup>3</sup> of  $\mathcal{W}$  is open.

Moreover, I claim that  $\mathcal{W}$  is a cover of  $X$ . Indeed, a point of  $x$  is either in  $A$  or not. If  $x$  is not in  $A$ , then  $x \in A^C$ , which is a member of  $\mathcal{W}$ . If  $x \in A$ , then there is some  $U \in \mathcal{U}$  for which  $x \in U$ . In particular,  $x$  is in the  $V$  we chose to correspond to  $U$ .

This shows that  $\mathcal{W}$  is an open cover.

By compactness of  $X$ ,  $\mathcal{W}$  admits a finite subcover. But if  $\mathcal{W}$  is an open cover of  $X$ , its pull-back to  $A$  is an open cover of  $A$ . Moreover, we see that  $i_A^{-1}(V_i) = V_i \cap A$  is precisely equal to some  $U_i \in \mathcal{U}$ . So this pullback is a finite subcover of  $\mathcal{U}$ , proving that  $A$  is compact.  $\square$

<sup>2</sup>As usual,  $\mathcal{T}_X$  stands for the topology on  $X$ .

<sup>3</sup>Member is a synonym for element. So a member of a set is the same thing as an element of a set.



## 10.5 Bonus: Relation to “bounded sequence” in analysis

**Remark 10.5.1** (Bounded sequences). In analysis, you may have heard about “bounded” sequences. This is a similar use of the word bounded. A sequence  $a_1, a_2, \dots$  is called bounded if there is some  $r > 0$  such that for every  $i$ ,  $a_i \in \text{Ball}(0, r)$ .

In real analysis, you also hear about sequences being bounded above or bounded below. We won’t use these notions as much in this course. Regardless, I’ll tell you what these mean. A sequence  $a_1, a_2, \dots$  is *bounded above* if there is some real number  $t$  such that, for every  $i$ ,  $a_i < t$ .

Likewise, a sequence  $a_1, a_2, \dots$  is *bounded below* if there is some real number  $s$  such that, for every  $i$ ,  $a_i > s$ .

Given a sequence  $a_1, a_2, \dots$ , the following are equivalent:

- (i) The sequence is bounded.
- (ii) The sequence is bounded above and bounded below.

I promise you can prove the equivalence of these two statements on your own. But now, we will ignore the notion of boundedness for sequence, and concentrate on the notion of boundedness in Definition 10.1.1.

## 10.6 Possible solutions

*Possible solution to Exercise 10.2.5.* Choose a point  $x \in \mathbb{R}^n$ . We must show that  $A = \{x\}$  is closed. There are two ways to see this. Any sequence  $a_1, a_2, \dots$  in  $A$  is a “constant” sequence, and hence converges to  $x$  (which is in  $A$ ). Thus  $A$  satisfies the “convergent sequence” criterion of closedness you proved in homework.

Another way to see this:  $A^c$  is equal to  $\mathbb{R} \setminus \{x\}$ . This can be written as a union of open intervals of finite radius, so is open.

As for  $[a, \infty)$ , let  $\mathcal{A}$  denote the collection of all pairs  $(x, r) \in \mathbb{R} \times \mathbb{R}_0$  so that the intersection of the open interval  $(x - r, x + r)$  and  $[a, \infty)$  is empty. Letting  $U_{(x,r)}$  be the open ball of radius  $r$  about  $x$ —also known as the open interval  $(x - r, x + r)$ —we see that the complement of  $[a, \infty)$  is the union

$$\bigcup_{(x,r) \in \mathcal{A}} \text{Ball}(x, r).$$

So  $[a, \infty)$  has open complement. The proof for  $(-\infty, a]$  is nearly identical.  $\square$

*Possible solution to Exercise 10.2.8.* (i). Because  $\emptyset$  is open (by definition of topology), and  $\emptyset^C = X$ ,  $X$  is closed. Likewise, because  $X$  is open (by definition of topology) and  $X^C = \emptyset$ , we see that  $\emptyset$  is closed. We've shown that  $X$  and  $\emptyset$  are elements of  $\mathcal{K}$ .

(ii). Let  $B_1, \dots, B_n$  be a finite collection of closed subsets. We must show that  $B_1 \cup \dots \cup B_n$  is closed. In other words, we must show that  $(B_1 \cup \dots \cup B_n)^C$  is open. By de Morgan's laws, we see

$$(B_1 \cup \dots \cup B_n)^C = B_1^C \cap \dots \cap B_n^C.$$

Since each  $B_i$  is closed by assumption, we know that each  $B_i^C$  is open. Thus the above is an intersection of finitely many open subsets of  $X$ . Such an intersection is known to be open by definition of topology.

(iii). Finally, given an arbitrary collection  $\{B_\alpha\}_{\alpha \in \mathcal{A}}$  of closed sets, we must show that the intersection is closed. That is, we must show that  $(\bigcap_{\alpha \in \mathcal{A}} B_\alpha)^C$  is open. Well, by de Morgan's laws,

$$\left( \bigcap_{\alpha \in \mathcal{A}} B_\alpha \right)^C = \bigcup_{\alpha \in \mathcal{A}} B_\alpha^C.$$

This last expression is a union of open subsets of  $X$ . By definition of topology, such a union is open.  $\square$

*Possible solution to Exercise 10.2.11.* If  $B \subset Y$  is closed, then  $B^C$  is open in  $Y$ . Because  $f$  is continuous,  $f^{-1}(B^C)$  is open in  $X$ . Moreover,

$$\begin{aligned} f^{-1}(B^C) &= \{x \in X \mid f(x) \in B^C\} \\ &= \{x \in X \mid f(x) \notin B\} \\ &= \{x \in X \mid x \notin f^{-1}(B)\} \\ &= (f^{-1}(B))^C. \end{aligned}$$

That is, the preimage of a complement is the complement of a preimage. So  $f^{-1}(B)$  has an open complement. By definition, this means  $f^{-1}(B)$  is closed.  $\square$

*Possible solution to Exercise 10.3.2.* To show  $\mathcal{U}$  is an open cover, we must show that it is a cover, and that every element of  $\mathcal{U}$  is open.

**$\mathcal{U}$  is a cover.** We must show that every  $x \in X$  is an element of some  $U \in \mathcal{U}$ . Given  $x$ , consider the point  $f(x) \in Y$ . We know that  $\mathcal{V}$  is a cover of  $Y$ , so there exists some  $V \in \mathcal{V}$  for which  $f(x) \in V$ . In particular,  $x \in f^{-1}(V)$  by definition of preimage.

On the other hand,  $U = f^{-1}(V)$  is an element of  $\mathcal{U}$  by definition of  $\mathcal{U}$ . So  $x \in U$ , and we are finished with this claim.

**$\mathcal{U}$  is an open cover.** We must show that every  $U \in \mathcal{U}$  is open. Well,  $U \in \mathcal{U}$  if and only if  $U = f^{-1}(V)$  for some  $V \in \mathcal{V}$  (by definition of  $\mathcal{U}$ ). And any  $V \in \mathcal{V}$  is known to be open (because  $\mathcal{V}$  is assumed to be an *open* cover). Because  $f$  is continuous, the pre-image of any open subset is open—in particular,  $U = f^{-1}(V)$  is open.  $\square$

*Possible solution to Exercise 10.4.2.* We must show that any open cover of  $f(X)$  admits a finite subcover.

By definition of subspace topology, a subset  $V \subset f(X)$  is open if and only if  $V = W \cap f(X)$  for some open subset  $W \subset Y$ . So, fixing an open cover  $\mathcal{V}$  of  $f(X)$ , we know that for every  $V \in \mathcal{V}$ , we may find some  $W \subset Y$  for which  $V = W \cap f(X)$ .

Let  $f' : X \rightarrow f(X)$  denote the continuous function guaranteed by the universal property of the subspace topology.<sup>4</sup> Because  $f'$  is continuous, we may pull back  $\mathcal{V}$  along  $f'$  and obtain an open cover of  $X$ . (This is Proposition 10.3.1.)

Concretely, this pull-back cover is

$$\mathcal{U} = \{U \subset X \mid U = (f')^{-1}(V) \text{ for some } V \in \mathcal{V}\}.$$

(As defined in Proposition 10.3.1.)

Now we use the fact that  $X$  is compact.  $\mathcal{U}$  is an open cover, so by compactness of  $X$ ,  $\mathcal{U}$  admits a finite subcover. Let us denote the elements of this finite subcover by

$$U_1, \dots, U_n.$$

Well, by definition, for each  $i$ , there is some  $V_i \in \mathcal{V}$  such that  $U_i = f^{-1}(V_i)$ . So consider the collection

$$V_1, \dots, V_n.$$

Note that each  $V_i$  is a subset of  $f(X)$ .

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<sup>4</sup>Remember, in terms of formulas,  $f'(x) = f(x)$ . The meat of  $f'$  is the ability to change the codomain of  $f$ .

I claim this collection  $\{V_1, \dots, V_n\}$  is a cover of  $f(X)$ . To see this, we must show that every element of  $f(X)$  is contained in some  $V_i$ . Well, let  $y \in f(X)$ . By definition,  $y$  is in the image of  $f$ , so there exists some  $x \in X$  for which  $f(x) = y$ . Because the  $U_1, \dots, U_n$  are a cover of  $X$ , we conclude that  $x \in U_i$  for some  $i$ . By definition, this  $U_i$  is the preimage of  $V_i$ , so we conclude that  $f(x) \in V_i$ . Recalling that we chose  $x$  so that  $f(x) = y$ , we conclude that the  $y$  we began with is an element of  $V_i$ . This shows that  $\{V_1, \dots, V_n\}$  forms a cover of  $f(X)$ .

So we have exhibited a finite subcover  $\{V_1, \dots, V_n\}$  of  $\mathcal{V}$ . □