Reading 7

Topological spaces and continuity

7.1 Some math philosophy

In mathematics, we often want to compare objects. For example, in previous classes, I think you learned that you can "compare" the sizes of different sets. And the main tool we have for comparing the sizes of sets is (perhaps counter-intuitively), the method of constructing *functions* between sets.[1]

For example, two sets have the same "size," or cardinality, if and only if there is a bijection between them.

7.1.1 Applying this philosophy to topology

In class, I've stated the slogan that "topology is the study of shapes." What is the appropriate way to compare different shapes?

7.1.2 Shapes, not sets

Perhaps the most important idea we must deal with is that shapes and sets are not the same thing. Certainly, shapes can be described as sets, but surely they have more structure than just a collection of elements.

[1] It was probably a large breakthrough in the history of mathematics when we realized that *functions* hold the key to comparing objects.

For example, $\mathbb{R}$ is an uncountable set. On the other hand, the open ball of radius 1 centered at the origin of $\mathbb{R}^2$ – which we write as $\mathrm{Ball}(0,1) \subset \mathbb{R}^2$ – is also uncountable. In fact, you can prove that these two sets are in bijection.[2] But "clearly," at least to our intuition, $\mathbb{R}$ and $\mathrm{Ball}(0,1)$ are very different looking shapes. $\mathbb{R}$ seems 1-dimensional, for example, while $\mathrm{Ball}(0,1) \subset \mathbb{R}^2$ seems 2-dimensional.

So a bijection alone can't tell us whether two shapes seem "equivalent" to us.

This is a game mathematicians play all the time. What *structure* should two things have that distinguish them?

In a bit of terminological tautology, this structure is called a *topology* on the set. We will define the word topology in Definition 7.2.1.

7.1.3 Continuous functions, not functions

And—just as with posets—once we have structures, we should ask for functions that respect these structures.

The functions that respect the topological structures will be called *continuous* functions. We will define what a continuous function is in Definition 7.5.1.

7.2 Key definitions

Definition 7.2.1. Let X be a set, and let $\mathcal{T}$ be a subset of the power set of X.[3] We call $\mathcal{T}$ a *topology on* X, or simply a *topology*, if the following holds:

1. Both $\emptyset$ and X are elements of $\mathcal{T}$.

2. For any collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of elements in $\mathcal{T}$, the union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

[2] Here is one proof: You can show that $\mathbb{R}$ is in bijection with the open interval $(0,1)$, so $\mathbb{R} \times \mathbb{R}$ is in bijection with $(0,1) \times (0,1)$. Meanwhile the open interval $(0,1)$ is in bijection with the open square $(0,1) \times (0,1)$; so composing these bijections, we see that $\mathbb{R}$ is in bijection with $\mathbb{R} \times \mathbb{R}$. On the other hand, we know that $\mathrm{Ball}(0,1)$ admits an injection from $\mathbb{R}$, while the cardinality of $\mathrm{Ball}(0,1)$ must be less than or equal to that of $\mathbb{R}^2$. Because $|\mathbb{R}| \leq |\,\mathrm{Ball}(0,1)| \leq |\mathbb{R}^2| = |\mathbb{R}|$ (here, the $|\mathbb{R}|$ refers to the cardinality of $\mathbb{R}$) we conclude that $\mathrm{Ball}(0,1)$ and $\mathbb{R}$ have the same cardinality.

[3] In other words, $\mathcal{T}$ is a collection of subsets of X.

is an element of $\mathcal{T}$.[4]

3. For any *finite* collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of elements in $\mathcal{T}$, the intersection

$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha$$

is an element of $\mathcal{T}$.[5]

Informally, $\mathcal{T}$ is picking out certain "special subsets" of X. So a topology on X is a choice of certain special subsets. Moreover, the totality of these special subsets must satisfy nice properties—the properties spelled out above.

This is one of the weirdest definitions that you've seen in your life. And rightly so. It's quite a whopper, and it's still one of the more surprising kinds of structures in mathematics.

But before we go on, let me just issue a big warning that a lot of students get confused by:

Warning 7.2.2. $\mathcal{T}$ is NOT a subset of X. It is a subset of $\mathcal{P}(X)$.

7.2.1 Open subsets

Before we go to the next example, let me emphasize something. The adjective "open" has been applied in this course to two different settings:

- When X is not just a set, but a *poset*, we know what it means for a subset of X to be "open."
- When $X = \mathbb{R}^n$, then we know what it means for a subset of X to be "open."

The same word is being used to describe two very different kinds of subsets—the first has to do with subsets that have some property with respect to a partial order relation, while the second example has to do with notions of open balls in Euclidean space. *Same word, different settings.*

The one commonality, though, is that the collection of open sets forms a *topology* in either example. In fact, the math community has adopted the following tradition:

[4] That is, if each U_α is in $\mathcal{T}$, then so is the union $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

[5] So if we have *finitely many* subsets $U_\alpha \subset X$ such that each U_α is in $\mathcal{T}$, then the intersection $\bigcap_{\alpha \in \mathcal{A}}$ is also in $\mathcal{T}$.

Definition 7.2.3 (Open subset)**.** Let X be a set, and let $\mathcal{T}$ be a topology on X. Then any member of $\mathcal{T}$ is called an *open* subset of X.

In other words, even though you are used to "open" having to do with the notion of "remove the boundary points"[6], today we have stepped into a world where "open" means something completely strange. Indeed, "open" has no meaning *until* somebody specifies a topology $\mathcal{T}$. Only then may you say whether a subset of X is open.

7.2.2 Topological spaces

We now know what a topology on a set is. It turns out that's all we need to know to understand what mathematicians call a "topological space:"

Definition 7.2.4. A pair $(X, \mathcal{T})$—of a set X, and a topology $\mathcal{T}$ on X—is called a *topological space*.

Remark 7.2.5. Similarly, if P is a set, and if $\leq$ is a partial order relation on a set P, the pair $(P, \leq)$ is called a *poset*. Of course, you have noticed me being lazy, and sometimes I just write "let P be a poset," with the relation $\leq$ being understood.

Likewise, I may often write "Let X be a topological space," with the choice of $\mathcal{T}$ to be understood, or implicit, or unspecified, in the notation.

7.3 Examples and exercises

Recall that a subset $U \subset \mathbb{R}^n$ is called *open* if U can be written as a union of open balls.

Theorem 7.3.1. Let $X = \mathbb{R}^n$ and let $\mathcal{T} = \{U \subset \mathbb{R}^n \,|\, U \text{ is open}\}$.[7] Then $\mathcal{T}$ is a topology on $\mathbb{R}^n$.

You know that your definition is sophisticated if it takes a *theorem* to produce examples. This is a good sign that we're onto something good! We will give the proof in Section 7.9.

[6] As in, $(-3, 3) \subset \mathbb{R}$ is an "open" interval, or the "open" ball $\text{Ball}(x, r) \subset \mathbb{R}^n$ is a disk, but with the boundary sphere of the disk removed.

[7] That is, $\mathcal{T}$ is the collection of open subsets of $\mathbb{R}^n$.

Definition 7.3.2. Let $X = \mathbb{R}^n$, and let $\mathcal{T}$ denote the collection of all subsets of $\mathbb{R}^n$ that can be expressed as unions of open balls. This $\mathcal{T}$ is called the *standard topology* on $\mathbb{R}^n$.

Exercise 7.3.3. Let $(P, \leq)$ be a poset. Recall from homework that a subset $U \subset P$ is called *open* if whenever p is an element of U, and whenever p' is some element of P satisfying $p \leq p'$, we can conclude that $p' \in U$.

Let $\mathcal{T} := \{U \subset P \,|\, U \text{ is open}\}$. In other words, we declare $\mathcal{T}$ to be the collection of all open subsets of P as defined in the previous paragraph.

Show that $\mathcal{T}$ is a topology on P.

Definition 7.3.4. The $\mathcal{T}$ above is called the *Alexandroff topology* on P.

Given a definition, you should always look for the easiest examples. Here are the two easiest kinds of topology on any set.

Definition 7.3.5. Let X be a set. The *trivial* topology on X is the topology $\mathcal{T} = \{\emptyset, X\}$.

The *discrete* topology on X is the topology $\mathcal{T} = \mathcal{P}(X)$.

Remark 7.3.6. In other words, the discrete topology is the topology for which *every* subset of X is considered open.

The trivial topology is the one in which only the empty set and X itself are considered open.

Exercise 7.3.7. Let X be a set. Prove that the discrete topology on X is a topology.

Let X be a set. Prove that the trivial topology on X is a topology.

7.4 Reminders on preimages

Let's have a quick reminder on what *preimages* are (Definition 1.9.1).

Let X, Y be sets, and let $f : X \to Y$ be a function. Fix also a subset $V \subset Y$. Then the *preimage* of V (under f) is defined to be

$$f^{-1}(V) := \{x \in X \,|\, f(x) \in V\}. \tag{7.4.0.1}$$

In words, the preimage of V is the collection of everything in x whose image lies in V.

Example 7.4.1. For example, if $f : \mathbb{R} \to \mathbb{R}$ is some function from calculus, and if $V = (0, \infty) \subset \mathbb{R}$, then $f^{-1}(V)$ is the set of all real numbers x such that $f(x) > 0$.

Warning 7.4.2. f^{-1} in (7.4.0.1) does *not* refer to a function called "f-inverse." This notation is confusing but unfortunately common – f^{-1} need not make sense on its own (because f might not be a bijection! So it might not have an inverse function). Instead, what makes sense is the full notation $f^{-1}(V)$ when V is a subset of the codomain.

Warning 7.4.3. $f^{-1}(V)$ is a *set*. It is not an element of X.

7.5 Continuous functions

As I mentioned in the beginning of this lecture, we should also ask what *kinds* of functions are deserving to be called maps of topological spaces. Such functions are called continuous, and we define them as follows. Note that the *only* structure we have at our disposal is a choice of $\mathcal{T}$, so that's all we can use in defining what a continuous map ought to be.

7.5.1 Definition

Definition 7.5.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.[8] A function $f : X \to Y$ is called *continuous* if preimages of open subsets are open.

Remark 7.5.2. In other words, f is called continuous if the following holds: For all open subsets $V \subset Y$, the preimage $f^{-1}(V)$ is also open.

Remark 7.5.3. Using other symbols, f is continuous if and only if $V \in \mathcal{T}_Y \implies f^{-1}(V) \in \mathcal{T}_X$.

Remark 7.5.4. Let's again use the tip that every definition consists of a *type* and a *condition*. The adjective "continuous" applies to what type? It applies to a function. In other words, the adjective "continuous" only describes functions. It doesn't describe sets, or spaces, or subsets; it only describes functions.

[8] I am using a subscript X in $\mathcal{T}_X$ to denote that $\mathcal{T}_X$ is a topology on X, while $\mathcal{T}_Y$ is a topology on Y.

This is one of the trippy things about topology. While everybody likes to think of $\mathbb{R}$ as a "continuum" or "a continuous thing," in math, we will almost never use the word continuous to describe a set like $\mathbb{R}$, we will only use the word continuous to describe *functions* between topological spaces.[9]

What is the condition we demand of the function (to call the function continuous)? If V is an open subset of the codomain, then the collection of all points that end up inside V is an open subset of the domain.

Exercise 7.5.5. Let X be a set and Y any topological space.

Show that any function $f : X \to Y$ is continuous if X is given the discrete topology.

Show that any function $f : Y \to X$ is continuous if X is given the trivial topology.

The following is an incredibly important property of continuous functions.

Proposition 7.5.6. Let X, Y, Z be topological spaces. Fix a continuous function $f : X \to Y$ and another continuous function $g : Y \to Z$. Then the composition

$$g \circ f : X \to Z$$

is also continuous.

Exercise 7.5.7. Prove Proposition 7.5.6.

Possible solution to Exercise 7.5.7. Let $W \subset Z$ be an open subset. We must prove that $(g \circ f)^{-1}(W)$ is an open subset of X. Observe:

$$\begin{aligned}(g \circ f)^{-1}(W) &= \{x \in X \mid (g \circ f)(x) \in W\} \\ &= \{x \in X \mid g(f(x)) \in W\} \\ &= \{x \in X \mid f(x) \in g^{-1}(W)\} \\ &= f^{-1}(g^{-1}(W)).\end{aligned}$$

But since g is assumed continuous, we know that $g^{-1}(W)$ is open. Hence $f^{-1}(g^{-1}(W))$ is open (because f is also assumed continuous).

This completes the proof. □

[9] Indeed, the notion of what we consider the "continuous" nature of $\mathbb{R}$ in real life actually has to do with the "least upper bound" property of $\mathbb{R}$, or the "complete" property of $\mathbb{R}$, both of which more or less tell us that there are no "holes" in $\mathbb{R}$.

7.6 Examples of continuous functions

The word "continuous" is now overloaded, in the sense that you have seen it in two different contexts—in this class, and also in calculus or analysis, where continuity had a different definition (presumably, a definition using ϵ-δ).

You will prove the equivalence of these two notions of continuity for homework. More precisely, you will show:

Theorem 7.6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and consider $\mathbb{R}$ as a topological space by equipping it with the standard topology.

Then the following are equivalent:

1. f is continuous (in the sense of this lecture).

2. For every x in the domain, and for every $\epsilon > 0$, there exists a $\delta > 0$ so that
$$|x' - x| < \delta \implies |f(x') - f(x)| < \epsilon.$$
(In other words, for every x' satisfying the inequality involving δ, we are guaranteed that $f(x')$ sastifies the inequality involving ϵ.)

Remark 7.6.2. This says that the notion of continuity (from topology) is identical to the notion of continuity (from calculus), at least when the function has domain and codomain equal to $\mathbb{R}$.

But to prove this theorem, you may want a few hints and tricks. The first hint is that you won't actually need to use the definition of absolute value, or the definition of distance—you will only need to use the fact that, setting

$$\text{dist}(x, x') = |x' - x|,$$

distance satisfies the triangle inequality:

$$\text{dist}(x, x') + \text{dist}(x', x'') \leq \text{dist}(x, x'').$$

You will also want to use the fact that a subset of $\mathbb{R}^n$ is open if and only if every element in it has "wiggle room." (Look on a previous lecture to find out what I mean by this.)

Theorem 7.6.3. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, both with the standard topology. Then any function that is a finite sum or product of the "standard" functions from calculus—polynomials in each coordinate, sin or cosine, et cetera—is continuous.

The proof of the above theorem isn't so bad, but I'll leave it as an extra credit assignment for future weeks. It's the kind of fact that many students would rather assume, so they can move on with their lives. So I won't dwell on it.

You may use the above theorem freely from now on.

Example 7.6.4. The following are all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$:

(a) $(x_1, x_2, \ldots, x_n) \mapsto x_1$.

(b) $(x_1, x_2, \ldots, x_n) \mapsto x_2$.

(c) For any i between 1 and n, the projection map $(x_1, x_2, \ldots, x_n) \mapsto x_i$.

(d) $(x_1, x_2, \ldots, x_n) \mapsto x_1^2 + x_2^2 + \ldots + x_n^2$.

(e) $(x_1, x_2, \ldots, x_n) \mapsto \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. (This is known as the "distance from the origin" function.)

(f) More generally, for any $y \in \mathbb{R}^n$, the function $x \mapsto \mathrm{dist}(x, y)$ is continuous. (This is the "distance from y" function.)

(g) $f(x_1, \ldots, x_6) = \sin(x_1)x^2 + \cos(x_4)e^{x_5} - \pi x^6$ is another example.

7.6.1 Continuous functions to $\mathbb{R}^n$

Finally, let me state another source of continuous functions:

Theorem 7.6.5. Let X be a topological space, and equip $\mathbb{R}^n$ with the standard topology. Then a function

$$f : X \to \mathbb{R}^n$$

is continuous if and only if the coordinate functions of f are continuous.

What do we mean by coordinate functions? Well, for every $x \in X$, $f(x)$ is determined by a finite collection of real numbers:

$$f(x) = (f_1(x), \ldots, f_n(x)).$$

Each number $f_i(x)$, $1 \leq i \leq n$, is of course a coordinate of $f(x)$. As we vary x, we see that each $f_i(x)$ determines a function $f_i : X \to \mathbb{R}$. These $f_1, \ldots, f_n$ are the coordinate functions of f.

So for example, a function from $\mathbb{R}$ to $\mathbb{R}^n$ is continuous if and only if each of its coordinate functions are continuous.

From hereon, you may assume the above two theorems (except for when you need to prove the first theorem in your upcoming homework).

7.7 Closed sets

Definition 7.7.1. Let X be a topological space with topology $\mathcal{T}$. We say that a subset $K \subset X$ is a *closed* subset if the complement of K is open. (That is, if $K^C \in \mathcal{T}$.)

Proposition 7.7.2. Let X, Y be topological spaces and $f : X \to Y$ a function. Then the following are equivalent:

1. f is continuous
2. The preimage of any closed set is closed.

Proof. Suppose f is continuous, and let $K \subset Y$ be closed. Then K^C is open, so $f^{-1}(K^C)$ is open. But $f^{-1}(K^C) = (f^{-1}(K))^C$, so (by definition of closed) we conclude that $f^{-1}(K)$ is closed.

Conversely, let $V \subset Y$ be open. We must show that $f^{-1}(V)$ is open. Well, by definition of closed, $V^C \subset Y$ is closed. Assuming (2), $f^{-1}(V^C)$ is closed. Because $f^{-1}(V^C) = (f^{-1}(V))^C$, we conclude that $f^{-1}(V)$ is open. This proves f is continuous. □

7.8 Homeomorphisms

So, when are two topological spaces equivalent?

Recall that for a poset, the notion of equivalence was exhibited by the notion of a poset isomorphism. A function $f : P \to Q$ is a poset isomorphism if

1. f is a map of posets,
2. f is a bijection[10], and
3. The inverse to f is also a map of posets.

The notion of an equivalence of topological spaces is called a *homeomorphism*, and is defined analogously:

Definition 7.8.1. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. f is called a *homeomoprhism* if

1. f is continuous
2. f is a bijection, and
3. The inverse to f is continuous.

7.9 Proof of Theorem 7.3.1

Proof. To prove $\mathcal{T}$ is a topology, we must prove all three properties outlined in Definition 7.2.1.

(1) First, we must show that $\emptyset$ and $\mathbb{R}^n$ are in $\mathcal{T}$. We saw in the lecture introducing open sets of $\mathbb{R}^n$ that $\emptyset$ is a union of open balls (it is the union of an *empty collection* of open balls).[11] Then, we also saw in that lecture that $\mathbb{R}^n$ itself may be expressed as a union of open balls, so $\mathbb{R}^n$ is indeed open.

(2) Now we must prove that an arbitrary union of open sets is again open. There are several ways to prove this, but let me prove it in a way that is not too notationally taxing.

Recall that $U \subset \mathbb{R}^n$ is open if and only if, for every $x \in U$, there exists a real number $r > 0$ for which $\text{Ball}(x, r) \subset U$. Now, if $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of open subsets, and if $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is their union, choose any $x \in U$. We must show that for some $r > 0$, $\text{Ball}(x, r) \subset U$. Well, since $x \in U$, we know there exists some $\alpha \in \mathcal{A}$ such that $x \in U_\alpha$. For any such U_α, we know that U_α is open, so there exists some $r > 0$ for which $\text{Ball}(x, r) \subset U_\alpha$. So we see

$$\text{Ball}(x, r) \subset U_\alpha \subset U$$

[10]this guarantees that f has an inverse

[11]Again, this is not an important point; you might just treat it as technically true. If you like, you may declare $\emptyset$ to be open by convention.

so $\text{Ball}(x, r) \subset U$, and we are finished with proving (2).

(3) We must prove that a *finite* intersection of open subsets of $\mathbb{R}^n$ is again an open subset. So let $U_1, \ldots, U_k$ be some finite collection of open subsets, and let $U = \bigcap_{i=1,\ldots,k} U_i$ be their intersection. Again, we must show that for any $x \in U$, there is some radius $r > 0$ such that the open ball $\text{Ball}(x, r)$ is a subset of U.

Well, since x is in the intersection of the U_i, we know that x is an element of U_i for all i. And because each U_i is open by hypothesis, we thus know that there is some number $r_i > 0$ so that $\text{Ball}(x, r_i)$ is a subset of U_i. So let's choose such real numbers $r_1, \ldots, r_k$. Note that if $r < r'$, then clearly $\text{Ball}(x, r) \subset \text{Ball}(x, r')$. So let r be the smallest real number among $r_1, r_2, \ldots, r_k$. Then $\text{Ball}(x, r) \subset \text{Ball}(x, r_i)$ for all i. So we can conclude that $\text{Ball}(x, r) \subset U_i$ for all i. Hence, $\text{Ball}(x, r)$ is a subset of the intersection U.

We have thus exhibited $r > 0$ for which $\text{Ball}(x, r) \subset U$, so U is open.

This finishes the proof of the theorem. □

Possible solution to Exercise 7.3.3. We must show that the Alexandroff topology indeed satisfies the three conditions laid out in Definition 7.2.1.

1. We saw as an example in homework that, indeed, the empty set and P itself are open.

2. You proved in homework that the union of open subsets of a poset is again open.

3. Finally, in homework, you proved something *stronger* then required of us: An intersection of *any* collection of open subsets is again open (not just finite collections).

□

7.10 What you are expected to know

You should be familiar with the following four very different-looking kinds of topological spaces:

- For any $n \geq 0$, $\mathbb{R}^n$ is a topological space (when equipped with the standard topology).

- For any poset P, P is a topological space (when equipped with the Alexandroff topology).
- For any set X, X is a topological space (when equipped with the discrete topology).
- For any set X, X is a topological space (when equipped with the trivial topology).

Remark 7.10.1. The last two examples highlight the need to specify the topology when the notation for X doesn't make it obvious. But rest assured that, usually, we study $\mathbb{R}^n$ as a space with the standard topology, and a poset P as a space with the Alexandroff topology – so we often won't specify the topologies on $\mathbb{R}^n$ and on a poset P in the notation.

You should know the definitions of:

1. Topology
2. Topological space
3. Open subset of a topological space
4. Closed subset of a topological space
5. Preimage
6. Continuous function
7. Homeomorphism

You are also expected to know various examples of continuous functions between Euclidean spaces (with their standard topologies). And you should know that continuous functions compose to be continuous (Proposition 7.5.6).