Reading 6

Closed sets, and practice

6.1 Miscellany

6.1.1 Definitions

Every definition is about specifying a name for a kind of object. It's important to look out for both the *type* of object being defined, and the *condition* imposed on that type for the definition to apply.

For example, a function is called a *bijection* if it is both an injection and a surjection. So, what is a bijection? The type of thing that a bijection is, is a function. The condition for a function to be a bijection is that the function be both injective and surjective.

Note that you would have no idea what a bijection is if you didn't understand the type (e.g., if you didn't know what a function is[1]) nor the conditions (e.g., if you didn't know what an injection or a surjection are).

6.1.2 Understanding some statements

"The crepuscular petrichor, so powerful, so vitalizing as to encourage even the Nebbish to ultracrepidarian pronouncements that would be ineffable to scholars even after their the most dedicated lucubrations, awakened his senses."

[1]And let me point out something: A function is not some thing you can always express like $f(x) = x^2 + 3x$. In calculus, this is more or less what "function" meant, but having taken more advanced math classes, you know that functions don't always have domain and codomain given by $\mathbb{R}$.

Most people have no idea what the above sentence means. But if it is our task to understand it, how would we? At the very least we should use a dictionary to look up the terms we don't understand.

In this class, the dictionary is your notes, or the class notes (posted in PDF format on the website). Sometimes, the dictionary is also the homework assignment (I have defined terms in homework assignment PDFs, and will continue to). Use these dictionaries.

And if you find that you need to look up almost every term in a sentence, that's fine. That's how it goes sometimes.

Half the battle of proving a statement is understanding the statement. Even before getting to the logical difficulties, you need to know what each of the terms means.

6.1.3 Where are you stuck?

The most important life skill that you will take with you beyond this course (if you work at developing it): The ability to identify where you are stuck.

Imagine you are a plumber, and a potential customer comes to you and says "My house needs a fix." You ask, okay, what about the house needs fixing? And the customer says "I don't know."

You would want more information, yes? At the same time, it may be completely reasonable for the customer (who is not a plumbing expert) to not know how exactly to articulate their problem. A frustrating experience for both!

When you are learning mathematics, or practicing any science, you are both the customer and the plumber. You will have a problem (a house) which you know you must solve or fix, yet have no idea how to go about it. But you *must* be able to identify what part of this issue is causing your confusion, or causing you to be stuck. If you don't know which pipe needs fixing, how are you going to fix it?

6.1.4 It's normal to be stuck

If all you have ever taken are algebra and calculus classes, you probably have a completely unrealistic expectation of how long a math problem takes to solve. This is why you could probably do dozens of math problems a day if you felt like it.

Especially are you are transitioning to proof-based mathematics, and especially in this class (where homework problems are difficult), this is an unrealistic expectation. I expect most of you to try to think about the lecture, or a problem, about an hour a day, and to not understand things for most of those minutes. When you "think about" the problem or the lecture notes, I expect you to be exploring the definitions, trying to understand examples, and also trying to produce examples yourself. It is this process of doing things on your own that is most important–exploring math is, to a mathematician, the same thing as a biologist finding specimen to study. You need to do it.

And, as with most sciences, exploration is about being lost, or at the very least, about wandering.

6.2 Review of complements

The nice thing about a class like this—which builds on previous classes—is that we can review some concepts.

Definition 6.2.1 (Complements)**.** Let A be a set, and $B \subset A$ a subset. Then the *complement* of B (inside A) is

$$A \setminus B := \{a \in A \,|\, a \notin B\}.$$

The complement is also sometimes denoted by

$$B^C \qquad \text{or} \qquad A - B.$$

In words, the complement of B is the set of all elements in A that are not in B.

Example 6.2.2. If $A = \mathbb{R}$ and $B = \mathbb{Q}$, then $A \setminus B$ is the set of all irrational numbers.

Example 6.2.3. If $A = \mathbb{R}$ and $B = (-5, 5)$, then $A \setminus B$ is the set of all real numbers whose absolute value is greater than or equal to 5.

6.3 Closed sets

You will explore this concept in your homework.

Definition 6.3.1. A subset $K \subset \mathbb{R}^n$ is called *closed* if the complement of K is open.

Remark 6.3.2. A minor remark about the use of 'if' in definitions. When *defining* a term, the word 'if' effectively means 'if and only if.' This is because we wouldn't call something "closed" if it weren't closed.

Note that *defining* a term is completely different from proving a statement about a term. If this remark confuses you, you should reach out to Hiro. This kind of mathematical language needs to be understood and used precisely!

Example 6.3.3. An example of a closed set is the complement of $(-5, 5)$ in $\mathbb{R}$. Another example of a closed set is the complement of an open ball in $\mathbb{R}^n$.

Note that—just as with open sets—it is important to specify, or be aware of, the "parent set" in which we are discussing closedness.

That's it. You'll have more practice in homework. Importantly, in homework, you'll have to utilize a new concept, called *convergence* of a sequence. As usual, keep in mind that proof takes a long time for most, so get started as soon as you can.

6.4 Practice

For the rest of today, I want you to work on the following exercises:

6.4.1 Review of Pre-4330 material

These exercises are optional, and are meant to give you practice with material you should have already seen before 4330. It's okay to forget some of the material over time, but you will immediately need to be able to do, and understand, these kinds of problems.

Exercise 6.4.1. Consider the sets

$$A = \{\text{Rosa, Sara, Tina}\} \qquad \text{and} \qquad B = \{\text{Rosa, Tina}\}.$$

(a) Write down every element of $A \times B$.

(b) Write down every element of $B \times A$.

(c) Exhibit a bijection between $A \times B$ and $B \times A$.

(d) Write down every element of $A \setminus B$.

Exercise 6.4.2. Let A be a set with 3 elements, and B a set with 4 elements. How many functions are there from A to B? How many functions are there from B to A?

Exercise 6.4.3. (a) Give an example of a finite set.

(b) Give an example of a countably infinite set.

(c) Give an example of an uncountably infinite set.

(d) True or False: $\mathbb{R}$ is countable.

(e) True or False: $\mathbb{R}^n$ is countable.

(f) True or False: $\mathbb{Q}$ is countable.

6.4.2 Posets

Exercise 6.4.4. How many functions are there from [1] to itself?

Of these, write down the functions that are *not* maps of posets.

Exercise 6.4.5. "Draw" the set $[2] \times [2]$. (Not as a poset; just draw the 3 x 3 array that you might normally draw for this product set.)

Then, draw the relation $\leq$. In other words, draw the subset of $[2] \times [2]$ associated to the relation $\leq$.

Exercise 6.4.6. Let A be a set. The *discrete* relation, or the *diagonal* relation, on A is the subset $\Delta_A \subset A \times A$ defined by

$$\Delta_A = \{(a, a)\}.$$

In other words, Δ_A consists of all ordered pairs $(a, b) \in A \times A$ for which$a = b$.

Prove that Δ_A defines a partial order relation. This is called the *discrete* partial order.

Exercise 6.4.7. (a) Give an example of a poset map $P \to Q$ that is a bijection, but is not an isomorphism of posets.

(Hint: What if $P = Q$ as sets, and the codomain is given a non-discrete partial order, but the domain is given a discrete partial order?)

(b) Let P be a poset. Is it possible that there is a poset map from P to itself, which is a bijection, but which is not an isomorphism of posets? (To be clear: Both the domain and codomain copies of P have the same poset structure.)

6.4.3 Open sets

Exercise 6.4.8. For each of the following, determine whether each of the following subsets is open or closed in $\mathbb{R}^n$. No need to write a proof. This is to give you experience in tinkering.

(a) $\mathbb{R}^n$ itself.

(b) $\emptyset$

(c) The open ball $\text{Ball}(x, r)$ centered at x, of radius $r > 0$.

(d) Fix two points $x, x' \in \mathbb{R}^n$ and two real numbers $r, r' > 0$. The union $\text{Ball}(x, r) \bigcup \text{Ball}(x', r')$. (If you haven't done this before: Make sure to draw a picture of what this looks like for various choices of x, x', r, r' in $\mathbb{R}^2$. The drawing won't help you answer whether union is open, but it is good practice.)

(e) $\mathbb{R}^2 \setminus \{(x_1.0)\}$, in $\mathbb{R}^2$.

(f) The set of all points $x \in \mathbb{R}^n$ of distance strictly less than 1 away from the origin.

(g) Fix a real number $a > 0$. The set $(-a, a) \times (-a, a) \times (-a, a) \subset \mathbb{R}^3$.

Exercise 6.4.9. Below, you will be given an indexing set $\mathcal{A}$ and a set $U_\alpha \subset \mathbb{R}^n$ for each $\alpha \in \mathcal{A}$. Determine if either of $\bigcup_\alpha U_\alpha$ or $\bigcap_\alpha U_\alpha$ is open in $\mathbb{R}^n$.

No need to write a proof. This is to give you experience in tinkering.

1. Let $\mathcal{A}$ be the set of positive real numbers, and for all $\alpha \in \mathcal{A}$, define U_α to be $\text{Ball}(0, \alpha)$ where $0 \in \mathbb{R}^n$ is the origin.

2. Let $\mathcal{A}$ be the set of positive real numbers, and for all $\alpha \in \mathcal{A}$, define U_α to be the set $(-\alpha, \alpha) \times \ldots \times (-\alpha, \alpha) \subset \mathbb{R}^n$. (Here, the direct product of open intervals is taken n times.)

3. Let $\mathcal{A}$ be the set of all elements of $\mathbb{R}^n$ having rational coordinates, and for all $\alpha \in \mathcal{A}$, let $U_\alpha = \mathrm{Ball}(\alpha, 1)$. (For a $x \in \mathbb{R}^n$ to have rational coordinates means that each $x_1, x_2, \ldots, x_n$ is a rational number).

4. Fix a point $x \in \mathbb{R}^n$. Let $\mathcal{A}$ be the set of all rational numbers, and for all $\alpha \in \mathcal{A}$, let $U_\alpha = \mathrm{Ball}(x, \alpha)$.

5. Let $\mathcal{A} \subset \mathbb{R}^n$ be the set of all n-tuples $(x_1, \ldots, x_n)$ such that $|x_1| + \ldots + |x_n| < 1$. For all $\alpha \in \mathcal{A}$, let $U_\alpha = \{\alpha\} \subset \mathbb{R}^n$ be the one-point set containing α.

6. Let $\mathcal{A}$ be the set of all positive real numbers. For all $\alpha \in \mathcal{A}$, let $U_\alpha = \mathbb{R} \times (-\alpha, \alpha) \subset \mathbb{R}^2$.

6.4.4 Looking ahead

Exercise 6.4.10. Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function sending x to (x, x). Suppose $V \subset \mathbb{R} \times \mathbb{R}$ is open. Prove that $f^{-1}(V)$ is open.

Exercise 6.4.11. Let $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function $(x_1, x_2) \mapsto x_1 + x_2$. Let $V \subset \mathbb{R}$ be open. Prove that $a^{-1}(V)$ is open.

Exercise 6.4.12. Let $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function $(x_1, x_2) \mapsto x_1 \cdot x_2$. Let $V \subset \mathbb{R}$ be open. Prove that $m^{-1}(V)$ is open.