Reading 5

Open sets in $\mathbb{R}^n$

5.1 Definitions

Let's talk a little about the idea of "definitions" in mathematics.

First, mammals. Give me some examples of mammals. You might say *human, dog, cat, kangaroo*. Okay, now let me ask: What exactly *is* a mammal? Can you give me precise criteria of what a mammal is, so that if I give your criteria to a computer, a computer could always and correctly identify something as a mammal given enough information?

In real life, we often operate by having some grasp of a word, without knowing precisely what it means. And, in the example of "mammal," we rarely try to *create* or *imagine* a completely new kind of mammal just to illustrate what the criteria provide for. But why did we even come up with the idea of a mammal? Probably because we noticed that there are certain animals that share many similarities (hair, mammary glands, non-egg birth) and so it was *convenient* to have a word for that.

Believe it or not, most definitions in math arise in a similar way. We often see a lot of examples around us, and we notice a commonality, and we decide to codify these examples under a single umbrella term. But, unlike biology, once the umbrella term and its criteria are delineated, we *can* imagine all kinds of crazy things so long as we can simply show that they fit the criteria. This is the power of math and of imagination.

In most math classes and math lectures, "definitions" are introduced first, and examples come later. Imagine if somebody began a biology lecture by saying "A *rhodophyte* is an aquatic eukaryotic alga with reddish coloring," but gave no examples; it may be hard for you to imagine what a rhodophyte is. Wouldn't you love a picture, or some examples, of rhodophytes?

In math class, you can also rest assured that the definitions we give are actually *useful*, just like it is incredibly useful to know about the idea that a classification of animal called "mammal" exists. But, unlike certain terms in biology, we cannot just immediately "see" an example of a math term. In fact, we often have to do some work to verify that certain things fit a mathematical definition. It is as though all biology definitions were based on traits that are only verifiable by intricate dissections, so that to verify a species is a mammal would necessitate time-intensive procedures.

5.2 Open subsets as a union of open balls

Let's recall the following:

Definition 5.2.1. Let $x \in \mathbb{R}^n$ and choose a real number $r > 0$. Then the *open ball of radius r centered at x* is

$$\mathrm{Ball}(x, r) := \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n} (y_i - x_i)^2 < r^2.\}$$

In words, $\mathrm{Ball}(x, r)$ is the set of all points y that have distance (strictly) less than r from x. Make sure you understand the truth of the previous sentence.

The following is a concept you will have to know very well in this class:

Definition 5.2.2 (Open subsets of $\mathbb{R}^n$)**.** A subset $V \subset \mathbb{R}^n$ is called *open* if A can be written as a union of open balls.

First, you should have a million questions. But let me just give a rephrasing of the definition: $V \subset \mathbb{R}^n$ is called *open* if there exists a set $\mathcal{A}$ and a function $\mathcal{A} \to \mathcal{P}(\mathbb{R}^n), \alpha \mapsto U_\alpha$ such that

1. U_α is an open ball for every $\alpha \in \mathcal{A}$, and
2. $V = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

Warning 5.2.3. The n matters in the definition of open. That is, $V \subset \mathbb{R}^n$ is open *as a subset of* $\mathbb{R}^n$. Openness is *not* an intrinsic property of the set V, and it matters that you are considering it as a subset of $\mathbb{R}^n$. Indeed, as we will see, an open subset in $\mathbb{R}^2$ tends to look very different from an open subset of $\mathbb{R}^3$.

5.3 The empty set is open

I'll begin with the most technical example, just so you're aware:

Example 5.3.1 (The empty set)**.** The empty set is an open subset of $\mathbb{R}^n$! You might think this is ludicrous, because it doesn't look like the union of any open balls. But, alas, consider the following. Let $\mathcal{A} = \emptyset$, and consider the function

$$\mathcal{A} \to \mathcal{P}(\mathbb{R}^n).$$

Note that I don't have to say anything about this function—there is a unique function from the empty set to any set. (This is a technicality, but also a wonderful fact.) Then

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \{x \in \mathbb{R}^n \,|\, \text{There is some } \alpha \in \mathcal{A} \text{ such that } x \in U_\alpha\}.$$

But there is *no* $\alpha \in \mathcal{A}$, so there are no x that fit the criteria above. In other words, this union contains *no* elements. That is, this union is the empty subset of $\mathbb{R}^n$.

If you did not like the above example, just take it as a given fact. The empty set is an open subset of $\mathbb{R}^n$.

Example 5.3.2 ($\mathbb{R}^n$)**.** Take a moment to show that $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.

5.3.1 Some proofs that $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$

Proof. Here is a potential proof: Let $\mathcal{A} = \mathbb{R}^n$ be the set of all elements of $\mathbb{R}^n$. Choose your favorite number, say 3. Then define

$$\mathcal{A} \to \mathcal{P}(\mathbb{R}^n), \qquad \alpha \mapsto \text{Ball}(\alpha, 3) =: U_\alpha.$$

Then

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is the set of all $x \in \mathbb{R}^n$ such that $x \in U_\alpha$ for some α. Well, for any $x \in \mathbb{R}^n$, let $x = \alpha$; we see that $x \in U_\alpha$ (because any point is certain within 3 units of itself). Thus $\mathbb{R}^n$ is a subset of this union; on the other hand, the union of subsets of $\mathbb{R}^n$ is always a subset of $\mathbb{R}^n$. So this shows that $\mathbb{R}^n$ is equal to this union of open balls. $\square$

This is of course not the only proof.

Proof. You could have chosen $\mathcal{A}$ to be the set of all n-tuplies $\alpha = (x_1, \ldots, x_n)$ such that each x_i is a rational number. Then

$$\mathcal{A} \to \mathcal{P}(\mathbb{R}^n), \qquad \alpha \mapsto \mathrm{Ball}(\alpha, 3)$$

would also exhibit $\mathbb{R}^n$ as a union of open balls. (This is because any $y \in \mathbb{R}^n$ is close enough to some x whose coordinates are rational.) $\square$

Proof. Another proof is as follows: Let $\mathcal{A} = \{1, 2, 3, \ldots\}$ be the set of positive integers. We'll write $n \in \mathcal{A}$ (instead of α). Choose a function as follows:

$$\mathcal{A} \to \mathcal{P}(\mathbb{R}^n), \qquad U_n = \mathrm{Ball}(0, n).$$

Then $\bigcup_{n>0} U_n$ is all of $\mathbb{R}^n$. $\square$

Remark 5.3.3. As you can see, there are *many* different ways to prove that $\mathbb{R}^n$ is open. The proofs always come down to finding a particular collection of open balls, and then proving that the union of that collection is $\mathbb{R}^n$ itself. This hails from the fact that you can write a given set as a union in many different ways. This non-uniqueness of proof will be very common as we move forward. (For some of you, this is what's great about math: There is more than one way to arrive at a solution!)

5.4 Other examples of open subsets

Example 5.4.1 (Open balls are open)**.** Let $U = \mathrm{Ball}(x, r) \subset \mathbb{R}^n$ be some open ball of radius r centered at $x \in \mathbb{R}^n$. This is an open subset of $\mathbb{R}^n$. (And *not* just because "open ball" is in the name—we have to *prove* that it's open as in Definition 5.2.2.)

Well, choose $\mathcal{A} = \{*\}$ to be a set with one element, and let

$$\mathcal{A} \to \mathcal{P}(\mathbb{R}^n), \qquad * \mapsto \mathrm{Ball}(x, r).$$

Then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \mathrm{Ball}(x, r)$. That is, any open ball is a union of a *single* open ball, and in particular, a union of open balls.

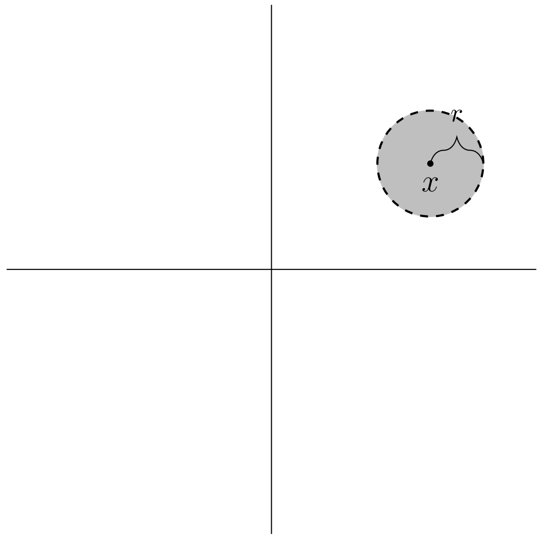

Figure 5.1: A picture of $\mathrm{Ball}(x,r) \subset \mathbb{R}^2$. (The dashed boundary means that the points of distance *exactly* r are not part of $\mathrm{Ball}(x,r)$.)

Example 5.4.2. Let $n = 2$, and let $x = (3,2), y = (3,3), z = (-2,1)$ be three points in $\mathbb{R}^2$. Consider the union

$$V = \mathrm{Ball}(x,1) \bigcup \mathrm{Ball}(z,2) \bigcup \mathrm{Ball}(y,1).$$

Because this is a union of (three) open balls, this is an open subset of $\mathbb{R}^2$. V is pictured in Figure 5.2. In fact, you can make all kinds of drawings by taking unions of open balls. Try picking a few random points for x, and a few random radii r, and drawing each $\mathrm{Ball}(x,r)$ (and their union).

Example 5.4.3. As the example of $\mathbb{R}^2$ illustrates, you can make infinitely large regions by taking unions of (infinitely many) open balls. Here is another such example. Let $V = \mathbb{R}^2 \setminus \{(x_1, 0)\}$. That is, let V be the set obtained by removing the x-axis from the xy-plane. This is drawn in Figure 5.3.

I must prove for you that V is open. Well, let $\mathcal{A} \subset \mathbb{R}^2 \times \mathbb{R}$ be the set of those pairs (x,r) such that (i) The coordinate x_2 of x is not zero, and (ii) $r < |x_2|$. Define

$$U_{(x,r)} = \mathrm{Ball}(x,r).$$

I leave it as an exercise to you that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = V$.

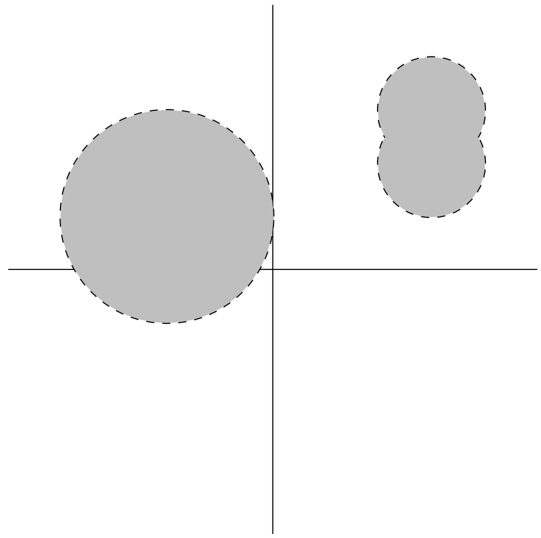

Figure 5.2: An image of the set $\mathrm{Ball}(x,1) \bigcup \mathrm{Ball}(z,2) \bigcup \mathrm{Ball}(y,1)$. Note again the dashed boundary.

Example 5.4.4. You might associate open balls with "round"-looking shapes. (After all, the boundary of a ball looks like a sphere, just scaled up or scaled down.) But you can make more "jagged-looking" shapes—really, the jaggedness is only an appearance, as the jagged parts are not part of the open set V we're about to make.

For example, we could have

$$V = \{(x_1, x_2) \mid |x_1| < 2, \qquad |x_2| < 3\} \subset \mathbb{R}^2.$$

And

$$W = \{(x_1, x_2) \mid x_2 > |x_1|\}.$$

V and W are both open subsets of $\mathbb{R}^2$. Can you write them as a union of open balls?

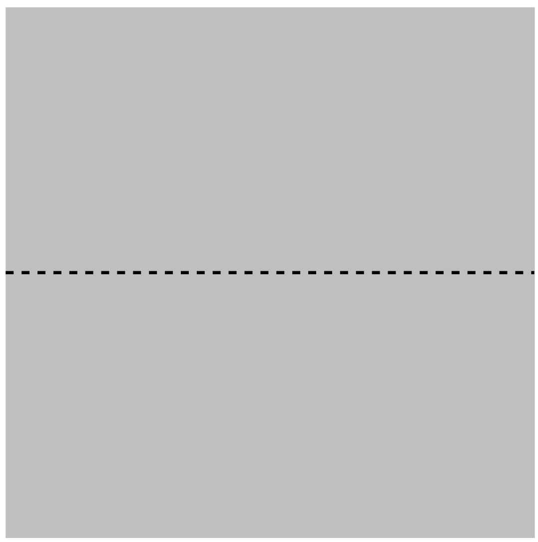

Figure 5.3: An image of the set $\mathbb{R}^2 \setminus \{(x_1, 0)\}$.

5.5 Subsets that are not open

Whenever you're given a new definition, you should think about things that *do not* fit the definition. So for example, are there subsets of $\mathbb{R}^n$ that are not open?

5.5.1 Using cardinality/size

In your previous proof class, you presumably learned about *cardinality*. Roughly speaking, the cardinality of a set is the size of the set. The big conceptual leap one has to make is the realization that there are many, many *infinite* cardinalities. For example, the set of integers is *countably* infinite[1], while the set of real numbers is *uncountably* infinite. And, as you showed in homework, $\mathcal{P}(\mathbb{R})$ is a set that is not in bijection with $\mathbb{R}$ (and in fact, has strictly larger cardinality than $\mathbb{R}$) so $\mathcal{P}(\mathbb{R})$ has a cardinality even larger than that of $\mathbb{R}$. Its size is another example of an uncountable cardinal.

Proposition 5.5.1. Let $n \geq 1$. Then for any $x \in \mathbb{R}^n$ and any $r > 0$, $\mathrm{Ball}(x, r) \subset \mathbb{R}^n$ is uncountably large.

Proof. For $n = 1$, this is because the open interval $(x-r, x+r)$ is uncountably large. (This open interval is in bijection with $\mathbb{R}$ itself, for example.)

For $n \geq 2$, let $x = (x_1, \ldots, x_n)$. Then the interval $(x_1 - r, x_1 + r)$ injects into $\mathrm{Ball}(x, r)$ by sending

$$y \mapsto (y, x_2, \ldots, x_n) \in \mathrm{Ball}(x, r) \subset \mathbb{R}^n$$

Since $\mathrm{Ball}(x, r)$ receives an injection from an uncountable set, it is at least uncountably large. $\square$

Corollary 5.5.2. If $V \subset \mathbb{R}^n$ is a non-empty, open subset, then V has uncountably many elements.

Proof. Since V is open, it can be written as a union of open balls. Because V is non-empty, this union must consist of at least one open ball, say $\mathrm{Ball}(x, r)$ for some $x, \in \mathbb{R}^n, r > 0$. But $\mathrm{Ball}(x, r)$ is uncountable (because of Proposition 5.5.1). Because V contains an uncountably large subset, V itself must be uncountably large. $\square$

[1] This means, by definition, that there is a bijection with the set of natural numbers

Thus, we conclude:

Corollary 5.5.3. If $A \subset \mathbb{R}^n$ is any non-empty subset with *countably* many elements, then A is not open.

Example 5.5.4 (A single-element subset is not open)**.** Assume $n \geq 1$ (note we exclude $n = 0$) and let $x \in \mathbb{R}^n$. Consider the set $B = \{x\} \subset \mathbb{R}^n$. ($B$ has only one element, called x.) Then B is finite, and in particular countable, so B could not be open.

Indeed, any finite subset of $\mathbb{R}^n$, for $n \geq 1$, is not open.

Example 5.5.5. Because $\mathbb{Q}$ is countable, $\mathbb{Q}$ is not an open subset of $\mathbb{R}$.

5.6 Open subset have "wiggle room" for each element

There are also plenty of uncountable sets that are not open. For example, the set of irrational numbers is a subset of $\mathbb{R}$ that is not open. The following lemma will help us think about what open subsets look like (see Figure 5.4):

Lemma 5.6.1. Let $\mathrm{Ball}(x, r) \subset \mathbb{R}^n$ be an open ball, and let $x' \in \mathrm{Ball}(x, r)$. Then there exists an $r' > 0$ so that $\mathrm{Ball}(x', r') \subset \mathrm{Ball}(x, r)$.

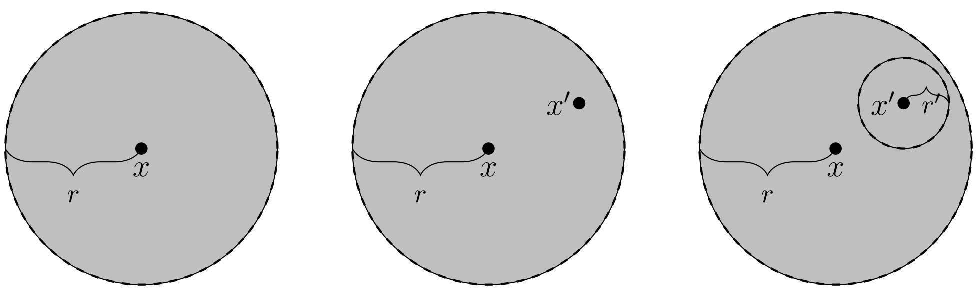

Figure 5.4: Given a point $x' \in \mathrm{Ball}(x, r)$, we can find an r' so that $\mathrm{Ball}(x', r') \subset \mathrm{Ball}(x, r)$.

One use of this lemma is to show that we can "re-center" an open ball, so that instead of having to deal with an open ball with center x, we can think about open balls centered at x', which may be psychologically easier to deal with.

Before we prove this Lemma, let's see one use of it:

Theorem 5.6.2. Let $V \subset \mathbb{R}^n$. The following are equivalent:

1. V is open. (See Definition 5.2.2.)
2. For every $x \in V$, there is a real number $r > 0$ so that $\text{Ball}(x, r) \subset V$.

In other words, $V \subset \mathbb{R}^n$ is open if and only if every point in V has some "wiggle room" inside of V—any point $x \in V$ can rest assured that there is some finite radius for which every point within that radius is still contained in V.

Proof of Theorem 5.6.2, assuming Lemma 5.6.1. (1) $\implies$ (2). V is open, so V may be expressed as a union $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ for some collection of open balls $U_\alpha = \text{Ball}(x_\alpha, r_\alpha)$. So if $x \in V$, by definition is union, there exists some α so that

$$x \in \text{Ball}(x_\alpha, r_\alpha).$$

By Lemma 5.6.1, we can find some r' so that $\text{Ball}(x, r') \subset \text{Ball}(x_\alpha, r_\alpha)$. By transitivity of inclusion (i.e., of the subset relation) we conclude

$$\text{Ball}(x, r') \subset V.$$

(2) $\implies$ (1). Let $\mathcal{A} = V$, and for every $\alpha = x \in \mathcal{A}$, let r_x denote some positive real number for which $\text{Ball}(x, r_x) \subset V$. (Such an r_x exists by assumption (2).) I claim that

$$V = \bigcup_{x \in \mathcal{A}} \text{Ball}(x, r_x).$$

Since each $\text{Ball}(x, r_x)$ is a subset of V, clearly this union is a subset of V. On the other hand, V is a subset of the union, as $x \in V \implies x \in \text{Ball}(x, r_x) \subset \bigcup_{x \in \mathcal{A}} \text{Ball}(x, r_x)$. This proves the claim. $\square$

Now let's prove the Lemma.

Proof of Lemma 5.6.1. Let $x' \in \text{Ball}(x, r)$. Let $d(x, x')$ denote the distance between x and x'. We know that $d(x, x') < r$ (because $x' \in \text{Ball}(x, r)$).

Then define $r' = r - d(x, x')$. This is a positive number because $d(x, x') < r$. Moreoever, if some point y is in $\text{Ball}(x', r')$, then I claim that y is also in $\text{Ball}(x, r)$.

In $\mathbb{R}$, or in $\mathbb{R}^2$, this follows from the triangle inequality. Remember that the triangle inequality tells us

$$d(x, x') + d(x', y) \geq d(x, y).$$

Well, we know that $d(x, x') < r$ and $d(x', y) < r' = d(x, x') - r$, so

$$d(x, x') + d(x', y) < d(x, x') + r' = d(x, x') + r - d(x, x') = r.$$

That is,

$$r > d(x, y).$$

Hence $y \in \text{Ball}(x, r)$ as well. This shows that $\text{Ball}(x', r') \subset \text{Ball}(x, r)$.

In $\mathbb{R}^n$, the triangle inequality still holds. To see this, given three points $x, x', y \in \mathbb{R}^n$, consider any plane that contains x, x', y. (If each of x, x', y are distinct, there is a unique such plane). Since the plane is a copy of $\mathbb{R}^2$ with the same notion of distance between points as $\mathbb{R}^2$ has (as determined by the Pythagorean theorem), the triangle inequality still holds on this plane, and hence in $\mathbb{R}^n$. □

5.7 The take-away

The main things you should take away from this lecture are:

(a) Theorem 5.6.2, which tells you that subset of $\mathbb{R}^n$ is open if and only if every point of the subset has some wiggle room. (That is, for every $x \in V$, there is some open ball centered around x that is contained inside V.)

(b) Not every subset of $\mathbb{R}^n$ is open. (For example, finite subsets, and more generally, countable subsets, are not open.)

(c) The proof of Lemma 5.6.1 (and hence Theorem 5.6.2) depended on the triangle inequality.