Reading 3

Partially ordered sets

Now for something totally different.

In life, useful sets are rarely "just" sets. They often come with additional structure. For example, $\mathbb{R}$ isn't useful just because it's some set. It's useful because we can add its elements, multiply its elements, and even *compare* its elements. If A is, for example, the set of all oranges and apples on earth, it's not as natural to do any of these things with elements of A.

Today, we'll talk about a structure called a *partial order*.

3.1 Preliminaries

First, let me remind you of something called the *Cartesian product,* or *direct product* of sets.

Definition 3.1.1. Let A and B be sets. Then the *Cartesian product* of A and B is defined to be the set consisting of all pairs (a, b) where $a \in A$ and $b \in B$. We denote the Cartesian product by

$$A \times B.$$

Example 3.1.2. $\mathbb{R}^2$ is the Cartesian product $\mathbb{R} \times \mathbb{R}$.

Example 3.1.3. Let T be the set of all t-shirts in Hiro's closet, and S the set of all shorts in Hiro's closet. Then $S \times T$ represents the collection of all possible outfits Hiro is willing to consider in August. (I.e., Hiro is willing to consider putting on any of his shorts together with any of his t-shirts.)

Note that we can "repeat" the Cartesian product operation, or apply it many times at once.

Example 3.1.4. $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Example 3.1.5. If F is the set of all flip-flops in Hiro's closet, then $F \times S \times T$ represents all possible outfits Hiro is willing to consider at the beach. Hiro actually never goes to the beach, but this is a math class where we disregard certain realities.

Remark 3.1.6. Note that $S \times T$ and $T \times S$ are *different* sets. The former set contains pairs (a, b) where the first element a must be in S, and the second element b must be in T.

In contrast, for (a, b) to be an element of $T \times S$, a must be in T and b must be in S.

If you regardless feel like $S \times T$ and $T \times S$ "feel" similar, you are correct. These two sets are naturally in bijection; they just are not the same set. Indeed, bijection is one of the first examples in math of the meaningful distinction between equality and equivalence. (The two sets are equivalent in a meaningful sense, but they are not the same in the most basic sense.)

3.2 Posets

So what is a poset?

Definition 3.2.1 (Informal)**.** Let P be a set. A *partial order* on P is a rule, called $\leq$. Given two elements $p, q \in P$, the rule lets us ask whether "$p \leq q$." This rule must satisfy the following:

1. (Reflexive) For every element p, we have that $p \leq p$.
2. (Antisymmetry) If $p \leq q$ and $q \leq p$, then $q = p$.
3. (Transitivity) If $p \leq q$ and $q \leq r$, then $p \leq r$.

The above definition is informal because the word "rule" is too imprecise. To make the definition more precise, we can note that the "rule" is some way of deciding something about pairs of elements of P. In other words, a rule is a way of deciding something about elements of $P \times P$. Well, what if we define a subset $R \subset P \times P$ as follows: If $p \leq q$, we declare that $(p, q) \in R$. (Note that this does not necessarily mean that $(q, p) \in R$.)

Then the above informal definition has a much more precise incarnation:

Definition 3.2.2 (Partial order)**.** Let P a set. A *partial order* on P is a choice of subset $R \subset P \times P$, satisfying the following:

1. (Reflexivity) For every $p \in P$, we have that $(p, p) \in R$.
2. (Antisymmetry) If $(p, q) \in R$ and $(q, p) \in R$, then $q = p$.
3. (Transitivity) If $(p, q) \in R$ and $(q, r) \in R$, then $(p, r) \in R$.

If R is a partial order on P, and if $(p, q) \in R$, we will write $p \leq q$.

Remark 3.2.3. This kind of refinement—of beginning with something that we want to define, but it being imprecise, and then having to refine it by being *clever* and redefining something in terms of weird set-theoretic ideas—happens all the time in math.

Remark 3.2.4. Fix a set A. It is incredibly common in math to utilize subsets $R \subset A \times A$. For this reason, any subset of $A \times A$ is often called a *relation* on A. For example, a partial order on A is sometimes called a partial order relation.

Later in this course, we will make use of another kind of relation, called equivalence relations.

Notation 3.2.5. Given a partial order relation $R \subset P \times P$, we will write the symbol $p \leq q$ to mean that $(p, q) \in R$. In words, $p \leq q$ will be read as "p is related to q," or sometimes, "p is less than or equal to q."

Example 3.2.6. Let's define $R \subset \mathbb{Z} \times \mathbb{Z}$ to be the set of all pairs (a, b) for which $b - a$ is not negative. Then $a \leq b$ has the usual meaning. In particular, $\leq$ is a partial order on $\mathbb{Z}$.

We can similarly define the usual "less than or equal to" partial order on $\mathbb{Q}$ and $\mathbb{R}$.

Warning 3.2.7. The symbol $\leq$ will be applied to any partial order, so I recommend that in such instances, you do *not* think of this as the usual less-than-or-equal-to for numbers. There are many partial orders that behave nothing like less-than-or-equal-to for numbers. For example, P may not even be a set of numbers!

Example 3.2.8 (The discrete partial order)**.** For any set P, define $R \subset P \times P$ to consist of the "diagonal"—that is, only elements of the form (p, p). Then R is a partial order, and $(P, \leq)$ is called a *discrete* poset. Two elements are related if and only if they are equal.

Remark 3.2.9. Whenever you are presented with a new idea in math, you should always try to understand the "cheapest," or the "easiest," or the "most trivial" example. Given any set A, it turns out you can find such a "cheapest" partial order; it is the discrete partial order.

Exercise 3.2.10. Let A be a set and (as usual) $\mathcal{P}(A)$ its power set. Define $R \subset \mathcal{P}(A) \times \mathcal{P}(A)$ to consist of those pairs (B, C) for which $B \subset C$.

Show that R defines a partial order on $\mathcal{P}(A)$.

Remark 3.2.11. Let P be a poset, and choose two elements $p, q \in P$. Then it is *not* true that $p \leq q$ or $q \leq p$—in other words, two elements may be "incomparable," or "not related."

This is illustrated in the previous example. For example, if $A = \{a, b\}$ is the two-element set, and $\mathcal{P}(A)$ the power set with partial order $\subset$, we see that $\{a\}$ and $\{b\}$ are not related (neither is a subset of the other).

Definition 3.2.12 (Posets)**.** A *partially ordered set* is a set equipped with a partial order. Because we are supremely lazy, we will call a partially ordered set a *poset*.

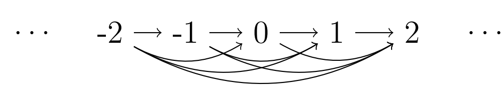
Notation 3.2.13. We will often denote a poset by the symbol $(P, \leq)$. We will also, due to laziness, just write "Let P be a poset," omitting the $\leq$ symbol.

If we are discussing two posets, we may refer to them as $(P, \leq_P)$ and $(Q, \leq_Q)$.

3.3 Visualizing posets

Let $(P, \leq)$ be a poset. We can "draw" the poset as follows: We just draw a point for every element of P, and draw an arrow from p to q if $p \leq q$.

As an example, here is a drawing of $(\mathbb{Z}, \leq)$ with the usual "less than or equal to" order:



This is very cluttered, so often we just draw the "shortest" arrows, when there is such a thing:

$$\cdots \quad \text{-2} \longrightarrow \text{-1} \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \quad \cdots$$

Example 3.3.1. It is impossible to draw $(\mathbb{R}, \leq)$ using "shortest" arrows. For example, if $a < b$, then you can always find some real number c so that $a < c < b$.

Example 3.3.2. Let A be a set with 1 element a. Then the poset $(\mathcal{P}(A), \subset)$ can be drawn as follows:

$$\emptyset \to A.$$

If A is a set with two elements a, b, then $(\mathcal{P}(A), \subset)$ can be drawn as follows:

$$\begin{array}{ccc} \{b\} & \longrightarrow & \{a,b\} \\ \uparrow & & \uparrow \\ \emptyset & \longrightarrow & \{a\} \end{array}$$

(Not drawn is the arrow from $\emptyset$ to $A = \{a, b\}$, as it is not a "shortest" arrow. However, drawing it shows how we can divide the above "square" into two triangles.)

Exercise 3.3.3. Draw the poset $(\mathcal{P}(A), \subset)$ when $A = \{a, b, c\}$.

Draw also the non-shortest arrows. Does this give you insight on whether you can fill up a cube with tetrahedra? How many tetrahedra?

3.4 More examples

Here are some facts that we won't verify, but you can verify in your free time, if you like:

Proposition 3.4.1. Let $(P, \leq)$ be a poset. We will denote as usual $R \subset P \times P$ the subset defining the partial order.

1. Choose any subset $A \subset P$. Define the relation $R_A = R \cap A \times A$. (In other words, for $a, b \in A$, we see that $a \leq_A b$ in A if and only if $a \leq b$ in P.) Then $(A, \leq_A)$ is a poset.

2. Let $(Q, \leq_Q)$ be a poset. Then define a poset structure on $P \times Q$ as follows:
$$(p, q) \leq (p', q') \iff p \leq p' \text{ and } q \leq q'.$$
Then $P \times Q$ is a poset.

In words, the above proposition says "subsets of posets are posets," and "products of posets are posets."

Definition 3.4.2. If $A \subset P$ is a subset of a poset, we call the poset structure from Proposition 3.4.1 the *subset poset structure*, or the *inherited partial order*, or the *inherited poset structure*.

If $P \times Q$ is a product of two posets, we call the poset structure from Proposition 3.4.1 the *product poset structure*.

Example 3.4.3. Fix an integer $n \geq 0$. Let
$$[n] := \{0, 1, \ldots, n\} \subset \mathbb{Z}.$$
Then $[n]$ inherits a poset structure from $\mathbb{Z}$. The picture of $[n]$ is as follows:
$$0 \to 1 \to 2 \to \ldots \to (n-1) \to n.$$

Example 3.4.4. Letting $[1]$ be the poset from Example 3.4.3 with $n = 1$, the set $[1] \times [1]$ is naturally a poset. We can draw it as follows:

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ \uparrow & & \uparrow \\ (0,0) & \longrightarrow & (1,0) \end{array}$$

Example 3.4.5. Let $S \subset \mathcal{P}(\{a, b\})$ be the subset containing every element of $\mathcal{P}(\{a, b\})$ except the empty set. Then we can draw S as follows:

$$\begin{array}{ccc} \{b\} & \longrightarrow & \{a,b\} \\ & & \uparrow \\ & & \{a\} \end{array}$$

Most posets are not easy to draw. For example, try drawing $S \times S$, where S is the poset from the previous example.

Here is an easier one to draw:

Exercise 3.4.6. Draw $[1] \times [1] \times [1]$ with the product poset structure.

Remark 3.4.7. When drawing posets, it might be helpful to just try to write out its elements, and then draw in some of the "shortest" arrows. It may not be obvious at first what the shortest arrow is, so you may need to give it a few tries.

3.5 Maps of posets

In previous classes, you've studied functions. In calculus, you studied functions of the form

$$f : \mathbb{R} \to \mathbb{R}.$$

And in your introduction to proof (or introduction to advanced math) class, you studied functions of the form

$$f : A \to B$$

where A and B are some sets.

Recall that the above notation conveys the following data:

1. f is a function.

2. The domain of f is A.

3. The codomain, or target, or f is B.

Well, whenever both A and B have similar structures, you can ask whether f *respects* those structure.

For example, let's suppose that A and B are both posets. (So they are equipped with partial order relations $\leq_A$ and $\leq_B$.) Then we could study not just arbitrary functions $f : A \to B$, but functions that "respect" the partial order. Let's define this notion:

Definition 3.5.1. Let $f : A \to B$ be a function, and A, B posets. We say that f is a *map of posets*, or a *poset map*, or a *poset morphism*, if the following is satisfied:

$$a_1 \leq_A a_2 \implies f(a_1) \leq_B f(a_2).$$

The sense in which f respects order is that if a_1 and a_2 are related (in A) before applying f, they are still related (in B) after applying f.

Remark 3.5.2. It may be that $a_1 \neq a_2$ and that $f(a_1) = f(a_2)$. So a map of posets need not be an injection. Note that $f(a_1) = f(a_2)$ does *not* imply that a_1 and a_2 are related in any way.

Example 3.5.3. Let A be any poset, and B a poset with a single element.[1] Then any function $f : A \to B$ is a map of posets.

Example 3.5.4. Suppose A is a poset, and let $S \subset A$ be a subset, endowed with the subset poset structure. Then the inclusion function $i : S \to A$ is a map of posets.

Exercise 3.5.5. Let A and B be posets, and let $A \times B$ be given the product poset structure. Notice that there are functions

$$p_A : A \times B \to A, \qquad p_A(a,b) = a, \qquad p_B : A \times B \to B, \qquad p_B(a,b) = b.$$

In words, the projection function p_A "forgets the B-coordinate," while p_B forgets the A-coordinate.

Show that both p_A and p_B are maps of posets.

3.6 Isomorphisms of posets

Definition 3.6.1. Let P and Q be posets, and let $f : P \to Q$ be a map of posets. We say that f is an *isomorphism of posets* if

1. f is a bijection, and

2. The inverse function to f is a map of posets.

When two posets are isomorphic, they behave identically. (Even though the two posets may not be the *same* poset.) Note that the first condition does not imply the second!

[1] Exercise: Any poset with a single element has a *unique* poset structure—the element is related to itself.

Example 3.6.2. Let $[1] = \{0 \leq 1\}$ with partial order $\leq$. Let $\leq_\delta$ denote the discrete partial order (so two elements are related if and only if they are equal—see Example 3.2.8).

Then the identity function $\text{id} : [1] \to [1]$ (which sends every element to itself) is a map of posets if you give the domain the discrete partial order $\leq_\delta$, and the target the usual partial order $\leq$. It is even a bijection.

However, it is not an isomorphism of posets.

Example 3.6.3. Let $A = \{a, b\}$ with $R = \{(a, a), (b, b), (a, b)\}$. (This is the partial order with $a \leq b$.)

Let $A' = \{a, b\}$ (the same set) with $R' = \{(a, a), (b, b), (b, a)\}$. (This is a partial order $\leq'$ with $b \leq' a$.)

Then the "identity function" $A \to A'$ sending $a \mapsto a$ and $b \mapsto b$ is *not* a map of posets. However, the "swap" function $a \mapsto b, b \mapsto a$ is an isomorphism of posets.

In terms of pictures, A could be drawn as

$$a \to b$$

while A' could be drawn as

$$a \leftarrow b.$$

Clearly these two pictures are not the same. However, if I swap the roles of a and b, the pictures become identical. This is what this poset isomorphism codifies.

Exercise 3.6.4. Exhibit an isomorphism of posets between $[1] \times [1]$ (with the product poset structure) and $\mathcal{P}(A)$, where $A = \{a, b\}$ is a set with two elements. (Here, $\mathcal{P}(A)$ is given the usual "subset" poset structure $\subset$.)

3.7 Bonus material: Every poset embeds into some power set

We've been exploring the notion of a partially ordered set, or poset for short. Posets are pedagogically useful because we can play with posets that are *finite* (like $[n]$, as opposed to $\mathbb{R}$) and at least they're somewhat intuitive. The easiest posets to think about are the $[n] = \{0 < 1 < \ldots < n\}$. And another natural poset is the power set $\mathcal{P}(A)$ of some set A.

In fact, *every* poset arises as a subposet of a power set.

Proposition 3.7.1. Let $(P, \leq)$ be a poset, and let $(\mathcal{P}(P), \subset)$ be the power set of P (with the usual "containment" partial order).

Then P is isomorphic, as a poset, to a subposet of $\mathcal{P}(P)$.

Exercise 3.7.2. Consider the function $j : P \to \mathcal{P}(P)$ that sends an element $q \in P$ to the set $j(q) = \{q' \mid q' \leq q\} \subset P$.

I claim that P is isomorphic (as a poset) to the subset $j(P)$—the image of j—endowed with the poset structure inherited from $\mathcal{P}(P)$.

Prove this, thereby proving Proposition 3.7.1.

Remark 3.7.3. Why is this Proposition useful? Well, you might be afraid that posets in general are allowed to look like crazy things. But the Proposition gives you some idea that the craziness can't be that bad. It says that no matter what your poset P is, it arises by choosing some power set $\mathcal{P}(A)$, and then choosing some elements of the power set. If you like, you can think of power sets like cubes (using the drawing trick we saw last time), and the Proposition says that every poset arises by simply choosing some vertices of the cube.

Remark 3.7.4. If you are familiar with abstract algebra, you can compare Proposition 3.7.1 to Cayley's Theorem in group theory. Just as every group has a natural embedding into its symmetric group, every poset has a natural embedding into its power set.

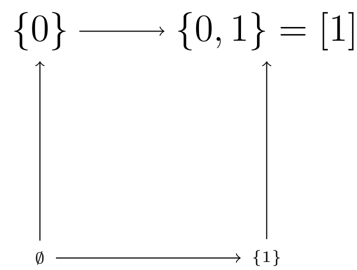
Example 3.7.5. Consider the poset $[1] = \{0 < 1\}$. The power set has four elements, and can be drawn as

$$\begin{array}{ccc} \{0\} & \longrightarrow & \{0,1\} = [1] \\ \uparrow & & \uparrow \\ \emptyset & \longrightarrow & \{1\} \end{array}$$

The morphism j from the exercise is a function $j : [1] \to \mathcal{P}([1])$. It sends

$$0 \mapsto \{a \mid a \leq 0\} = \{0\}, \qquad 1 \mapsto \{a \mid a \leq 1\} = \{0, 1\}.$$

So [1] can be found inside $\mathcal{P}([1])$ as follows:



(I've indicated the image of j using bigger font.)

Example 3.7.6. You should work out how [2] sits inside the cube $\mathcal{P}([2])$ as the elements $\{0\}, \{0,1\}, \{0,1,2\}$.